PROBLEM SET 3

1. [Simon's-type inequality] Let $u \geq 0$ satisfy

$$-\Delta u \leq Cu^2 \quad \text{in } B_1 \subset \mathbb{R}^n.$$

Assume that

$$\int_{B_1} u^{\frac{n}{2}} \leq \varepsilon$$

for $\varepsilon > 0$ sufficiently small. Show that

$$\sup_{B_{1/2}} u \leq C.$$

(This inequality is a Simon's type identity satisfied by the second fundamental form of minimal minimal surfaces. Such conclusion of small energy implying regularity is called an ε - regularity theorem.)

Hint: use test function of the form $\zeta^2 u^\beta$ for some compactly supported $\zeta \in C_0^1(B_1), \beta > 0$ and do integration by parts as in the lecture notes.

2. [Small energy harmonic map is constant] Prove that there exists $\varepsilon > 0$ such that if $u : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ satisfies the harmonic map equation

$$\Delta u = -|\nabla u|^2 u$$

and has small energy

$$\int_{\mathbb{R}^2} |\nabla u|^2 < \varepsilon,$$

then u is constant.

Hint: Deduce an differential equation / inequality for $v = |\nabla u|^2$ and apply Moser iteration technique.