PROBLEM SET 3

1. [Continuity method] Let $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ be an elliptic operator with Hölder continuous coefficients and $c \le 0$, and $U \subset \mathbb{R}^n$ be a bounded smooth domain. Then the following 2 statements are equivalent:

- The Dirichlet problem
$$\begin{aligned} \Delta u =& f, \quad \text{in } U \\ u =& \phi, \quad \text{on } \partial U \end{aligned}$$
has a unique solution $u \in C^{2,\alpha}(U)$, for any $f \in C^{0,\alpha}(U)$ and $\phi \in C^{2,\alpha}(\partial U)$.
- The Dirichlet problem
$$\begin{aligned} Lu =& f, \quad \text{in } U \\ u =& \phi, \quad \text{on } \partial U \end{aligned}$$
has a unique solution $u \in C^{2,\alpha}(U)$, for any $f \in C^{0,\alpha}(U)$ and $\phi \in C^{2,\alpha}(\partial U)$.

Hint: Define a family of operators $L_t = t\Delta + (1-t)L, \forall t \in [0,1]$, show that the map $T(u) =: L_s^{-1}(f) + L_s^{-1}[L_s - L_t](u)$ has a fixed point if L_s is invertible.

2. [Existence of principle eigenfunctions] Let $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ be an elliptic operator with Hölder continuous coefficients, and $U \subset \mathbb{R}^n$ be a bounded smooth domain. Consider the Dirichlet eigenvalue problem
$$\begin{cases} Lu + \mu u = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

- Show that there exists a smallest $\mu_1 \in \mathbb{R}$ ((principal eigenvalue) and $u_1 \in H_0^1(U)$, $u_1 \not\equiv 0$, such that
$$Lu_1 + \mu_1 u_1 = 0 \quad \text{in } U.$$
- Show that the eigenfunction u_1 can be chosen so that
$$u_1 > 0 \quad \text{in } U.$$
- Show that the first eigenvalue μ_1 is simple, i.e. the space of solutions to
$$Lu + \mu u = 0, \qquad u|_{\partial U} = 0,$$
is one–dimensional.
- Show that any eigenfunction corresponding to an eigenvalue $\mu > \mu_1$ must change sign in U.

3. Let u satisfies the minimal surface equation in $U \subset \mathbb{R}^n$ with boundary condition $\phi \in C^\infty(\partial U)$, where U is a smooth bounded open domain. Knowing that there is uniform gradient bound for $\|\nabla u\|$ in U. Show that $\partial_k u$ also satisfies and elliptic PDE for each $k = 1, \dots n$. And thus by maximum principle
$$\sup_U |\nabla U| \le C \sup_{\partial U} |\nabla U|.$$