PROBLEM SET 2

1.

- Using the half space Liouville theorem from Problem set 1, prove the following boundary Schauder estimate for Laplace operator (again by Simon's blow up argument): Denote by $B_R^+ = B_R \cap \{x_n > 0\} \subset \mathbb{R}^n$ and $S_R = B_R \cap \{x_n = 0\}$. Let $u \in C^2(B_{2R})$ satisfies
$$\begin{aligned} \Delta u =& f \quad \text{in } B_{2R} \\ U =& \phi \quad \text{on } S_{2R}, \end{aligned}$$
then
$$\|\mathrm{Hess}_u\|_{C^{0,\alpha}(B_R)} \leq C(\|f\|_{C^{0,\alpha}(B_{2R})} + \|\phi\|_{C^{2,\alpha}(S_{2R})}).$$
- Using the boundary Schauder estimate above and fixed coefficient argument as in the lecture to prove a global Schauder estimate (up to the boundary).

2. Let (K, d) be a compact metric space.

- (Arzelà–Ascoli) Let $\mathcal{F} \subset C(K)$. The family $\mathcal{F}$ is called *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,
$$d(x,y) < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \qquad \text{for all } f \in \mathcal{F}.$$
Prove that $\mathcal{F}$ is relatively compact in $C(K)$ if and only if it is uniformly bounded and equicontinuous.
- (Compact Hölder embedding) Let $0 < \beta < \alpha \leq 1$. Prove that the embedding
$$C^{0,\alpha}(K) \hookrightarrow C^{0,\beta}(K)$$
is compact.
- If K is not assumed to be compact, is the embedding
$$C^{0,\alpha}(K) \hookrightarrow C^{0,\beta}(K)$$
still true?

3. (Interpolation inequality) For any $\varepsilon \in (0,1)$ there exists C_ε so that the following holds:
$$\|u\|_{C^2(B_1)} \leq \varepsilon \|u\|_{C^{2,\alpha}(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}.$$
Hint: Use a blow up / contradiction argument and assume to the contrary
$$\|u_m\|_{C^2(B_1)} > \varepsilon \|u_m\|_{C^{2,\alpha}(B_1)} + m\|u_m\|_{L^\infty(B_1)}, m \to \infty.$$