PROBLEM SET 1

1. Suppose $u \in C^2(U)$ is a harmonic function on $B_r(x_0) \subset R^n$, use the idea of proof for gradient bound to show that

$$D^\alpha u(x_0) \leq \frac{C}{r^{n+k}} \|u\|_{L^1(B_{x_0}(r))},$$

with the norm of multiindex α being $|\alpha| = k$.

2. Suppose $u \in C^2(B_1), u \geq 0$ is a non-negative harmonic function.

- Use mean value theorem to prove the Harnack inequality

$$\sup_{B_{\frac{1}{2}}} \leq C(n) \inf_{B_{\frac{1}{2}}}.$$

- Using the Harnack inequality to show the following oscillation decay

$$\mathrm{osc}_{B_r} u \leq C r^\alpha \mathrm{osc}_{B_1},$$

where

$$\mathrm{osc}_U u = \sup_U u - \inf_U u.$$

In fact, the oscillation decay implies that $u \in C^{0,\alpha}$.

3. Prove the half space Liouville theorem for harmonic functions with Dirichlet boundary data: Let $C < \infty, \varepsilon > 0$. Denote by $\mathbb{H}^n = \{x_n > 0\} \subset \mathbb{R}^n$ and $B_r^+(0) = B_r^(0) \cap \mathbb{H}^n$. If $u : H^n \to \mathbb{R}$ is a harmonic function satisfying the boundary condition

$$u|_{x_n=0} = 0$$

and the growth bound

$$\sup_{B_r^+(0)} |u| \leq C r^{k-\varepsilon}$$

for some positive integer $k \in \mathbb{N}$, then u must be a polynomial of degree at most $k-1$.

4. (Challenge) Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is an entire k-harmonic function, namely

$$(\Delta)^k u = 0, \quad \text{for some } k \in \mathbb{N}.$$

If u satisfies the following growth bounds

$$\limsup_{r\to\infty} \frac{1}{r^n} \int_{B_{2r}\setminus B_r} \Delta u \leq 0$$

$$\limsup_{r\to\infty} \frac{1}{r^n} \int_{B_{2r}\setminus B_r} |\nabla u|^2 \leq 0,$$

then u must be a constant.