LECTURE 5: ε - REGULARITY THEOREM AND PARTIAL REGULARITY FOR HARMONIC MAPS

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Let $u : \Omega \subset \mathbb{R}^n \to (N, h)$ be a map into a compact Riemannian manifold N (isometrically embedded in $\mathbb{R}^k$). The *Dirichlet energy* is

$$E(u;\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx,$$

and its critical points under compactly supported variations are the *harmonic maps*.

Let $N \subset \mathbb{R}^k$ be a Nash embedding. Consider a smooth family of variations $u_s : \Omega \to N$ with $u_0 = u$ and the first variation field

$$\phi(x) := \frac{d}{ds}\bigg|_{s=0} u_s(x) \in T_{u(x)}N$$

must be tangential because the maps u_s all have images in N.

Critical points of the energy satisfy

$$\begin{aligned} 0 &= \frac{d}{ds}\bigg|_{s=0} E(u_s;\Omega) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{1}{2}\int_\Omega |\nabla u_s|^2\, dx \\ &= \int_\Omega \langle \nabla u, \nabla \varphi\rangle\, dx \\ &= -\int_\Omega \langle \Delta u, \varphi\rangle\, dx, \end{aligned}$$

for all $\phi \in C_c^\infty$.

This implies that $\Delta u(x)$ is orthogonal to $T_{u(x)}N$ for a.e. x, i.e.

$$\Delta u(x) = \Delta u(x)^\perp = \sum_{i=1}^n \left(\nabla^{\mathbb{R}^k}_{\partial_i}\partial_i u\right)^\perp = A(u)(\nabla u, \nabla u)$$

Thus the Euler–Lagrange equation for critical points of the Dirichlet energy under variations through maps into N (the *harmonic maps*) is

$$\Delta u = A(u)(\nabla u, \nabla u)\ . \tag{0.1}$$

Let $u : \Omega \subset \mathbb{R}^n \to N$ be a smooth harmonic map, and set

$$e := |\nabla u|^2.$$

Then the Bochner identity for harmonic maps gives

$$\Delta e = 2|\nabla^2 u|^2 - 2\big\langle R^N(\nabla u, \nabla u)\nabla u, \nabla u\big\rangle, \tag{0.2}$$

where R^N denotes the Riemann curvature tensor of the target manifold N.

Since N is compact, its curvature is bounded, and hence

$$-\Delta e \le C\, e^2 \qquad \text{in } \Omega, \tag{0.3}$$

for a constant $C > 0$ depending only on the geometry of N. The inequality (0.3) is understood in the weak (distributional) sense for $u \in W^{1,2}$.

1. Epsilon-regularity

Theorem 1.1 (Epsilon-regularity in dimension 2)**.** *There exists $\varepsilon_0 > 0$ such that the following holds. If $u : B_1 \subset \mathbb{R}^2 \to N$ is a weak harmonic map satisfying*

$$\int_{B_1} |\nabla u|^2 \le \varepsilon_0,$$

then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C \left(\int_{B_1} |\nabla u|^2 \right)^{1/2},$$

where C depends only on the geometry of N. In particular, $u \in C^\infty(B_{1/2})$.

Proof. We want to apply Moser iteration to (0.2)

Fix $0 < \rho < R \le 1$, and let $\eta \in C_c^\infty(B_R)$ satisfy

$$\eta \equiv 1 \text{ on } B_\rho, \qquad 0 \le \eta \le 1, \qquad |\nabla \eta| \le \frac{2}{R - \rho}.$$

For $p \ge 1$, multiply (0.2) by $\eta^2 e^{p-1}$ and integrate by parts, we get

$$\int \Delta e \eta^2 e^{p-1} \le \int C\eta^2 e^{p+1}$$

$$\int 2\eta e^{p-1}\langle \nabla e, \nabla \eta\rangle + \int (p-1)\eta^2 e^{p-2}|\nabla e|^2 \le \int C\eta^2 e^{p+1}$$

$$-2\int |\nabla e||\nabla \eta|\eta e^{p-1} + \int (p-1)\eta^2 e^{p-2}|\nabla e|^2 \le \int C\eta^2 e^{p+1}$$

$$-\int \frac{2}{p-1}|\nabla \eta|^2 e^p - \int \frac{p-1}{2}\eta^2 e^{p-2}|\nabla e|^2 + \int (p-1)\eta^2 e^{p-2}|\nabla e|^2 \le \int C\eta^2 e^{p+1}$$

Namely

$$\int |\nabla(\eta e^{p/2})|^2 \le C \int |\nabla \eta|^2 e^p + C \int \eta^2 e^{p+1}. \tag{1.1}$$

By Sobolev embedding $W^{1,2} \mapsto L^q$ compactly for $q \le \frac{2n}{n-2}$, we have

$$\|e\|_{L^{\frac{p}{2}\cdot\frac{2n-\delta}{n-2}}(B_r)} \le C_\eta \|e\|_{L^{p+1}(B_R)}.$$

In dimension $n = 2$, this directly implies L^q bound of u for any $q > 1$
In dimension $n \geq 3$, by choosing $\delta < \frac{4p+4-2n}{p}$ we get an improvement as

$$\frac{p}{2} \cdot \frac{2n-\delta}{n-2} > p+1.$$

Moser iteration then implies L^∞ bound of e by L^{p+1} bound of e. (Although the energy bound only implies L^1 norm of e, so we don't a priori have L^{p+1} bound of u to get L^∞ by the Dirichlet energy.)

□

This can be generalized to the case of minimizing harmonic maps in higher dimensions.

Theorem 1.2 (Schoen–Uhlenbeck ε–regularity). *Let $n \geq 3$ and let N be a compact Riemannian manifold. There exist constants $\varepsilon_0 > 0$ and $C > 0$, depending only on n and the geometry of N, such that the following holds.*

If $u \in W^{1,2}(B_r(x_0), N)$ is an energy-minimizing harmonic map *and*

$$r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \leq \varepsilon_0, \tag{1.2}$$

then u is smooth in $B_{r/2}(x_0)$ and satisfies the estimate

$$\sup_{B_{r/2}(x_0)} |\nabla u| \leq C\, r^{-1} \left(r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \right)^{1/2}. \tag{1.3}$$

2. Estimate on the size of singular set and partial regularity

Using the epsilon regularity result, we say that the energy minimizing harmonic maps are regular away from a measure 0 set. Indeed, one can estimate the Hausdorff dimension of the singular set.

Theorem 2.1 (Schoen–Uhlenbeck partial regularity). *Let $n \geq 3$ and let N be a compact Riemannian manifold. Suppose that*

$$u \in W^{1,2}(\Omega, N)$$

is an energy-minimizing harmonic map. *Then there exists a closed set*

$$\Sigma \subset \Omega$$

such that

(1) $u \in C^\infty(\Omega \setminus \Sigma)$*;*
(2) *the singular set Σ has Hausdorff dimension at most $n-2$, i.e.*

$$\dim_{\mathcal{H}}(\Sigma) \leq n-2.$$

Proof. An important tool we use is the notion of energy density and the monotonicity formula.

For $x \in \Omega$ and $r > 0$ with $B_r(x) \subset \Omega$, define the scaled energy

$$\theta(x,r) := r^{2-n} \int_{B_r(x)} |\nabla u|^2.$$

Since u is an energy-minimizing (hence stationary) harmonic map, the monotonicity formula implies that $\theta(x,r)$ is nondecreasing in r. Therefore the limit

$$\theta(x,0^+) := \lim_{r \downarrow 0} \theta(x,r)$$

exists for every $x \in \Omega$.

Let $\varepsilon_0 > 0$ be the constant from the ε–regularity theorem and define the singular set

$$\Sigma := \left\{ x \in \Omega : \theta(x,0^+) \geq \varepsilon_0 \right\}.$$

By the ε–regularity theorem, u is smooth in $B_{r_0/2}(x_0)$. Hence

$$u \in C^\infty(\Omega \setminus \Sigma).$$

It is not hard to see that the regular set is open and the singular set is closed.

Now fix a compact set $\Omega' \Subset \Omega$. For each $x \in \Sigma \cap \Omega'$, by definition of Σ and monotonicity, there exists $r_x > 0$ such that $B_{5r_x}(x) \subset \Omega$ and

$$\int_{B_{r_x}(x)} |\nabla u|^2 \geq \varepsilon_0 \, r_x^{n-2}. \tag{2.1}$$

The family $\{B_{r_x}(x)\}_{x \in \Sigma \cap \Omega'}$ is a covering of $\Sigma \cap \Omega'$. By the Vitali covering lemma, there exists a countable subcollection $\{B_{r_i}(x_i)\}$ such that

(1) the balls $B_{r_i}(x_i)$ are pairwise disjoint,
(2)
$$\Sigma \cap \Omega' \subset \bigcup_i B_{5r_i}(x_i).$$

So by (2.1) and the disjointness of the balls,

$$\varepsilon_0 \sum_i r_i^{n-2} \leq \sum_i \int_{B_{r_i}(x_i)} |\nabla u|^2 = \int_{\cup_i B_{r_i}(x_i)} |\nabla u|^2 \leq \int_\Omega |\nabla u|^2 < \infty.$$

Therefore

$$\sum_i r_i^{n-2} \leq \frac{1}{\varepsilon_0} \int_\Omega |\nabla u|^2.$$

Using the covering by $B_{5r_i}(x_i)$ and the definition of Hausdorff measure, we conclude

$$\mathcal{H}^{n-2}(\Sigma \cap \Omega') \leq C(n) \sum_i (5r_i)^{n-2} \leq \frac{C(n)}{\varepsilon_0} \int_\Omega |\nabla u|^2 < \infty.$$

Since $\Omega' \Subset \Omega$ was arbitrary, it follows that $\mathcal{H}^{n-2}(\Sigma)$ is locally finite in Ω, and hence

$$\dim_{\mathcal{H}}(\Sigma) \leq n-2.$$

□

3. Harmonic maps from $\mathbb{S}^2$ and minimal spheres

Let N be a closed Riemannian manifold. Define

$$E_{\min}(N) := \inf \Big\{ E(\omega) : \ \omega : \mathbb{S}^2 \to N \text{ nonconstant harmonic map} \Big\} \in (0, \infty].$$

The key fact is that the ε–regularity threshold in dimension two is controlled by $E_{\min}(N)$.

Theorem 3.1. *Let $0 < \varepsilon_0 < E_{\min}(N)$. Then there exists $C = C(N, \varepsilon_0)$ such that:*

If $u : D \subset \mathbb{R}^2 \to N$ is harmonic and

$$E(u; D) = \frac{1}{2} \int_D |\nabla u|^2 \leq \varepsilon_0,$$

then

$$\sup_{D_{1/2}} |\nabla u| \leq C \Big(\int_D |\nabla u|^2 \Big)^{1/2},$$

and in particular u is smooth on $D_{1/2}$.

We see that singular points have energy concentration, and for harmonic maps from $\mathbb{S}^2$, these energy concentration points are characterised by bubbling.

Theorem 3.2 (Bubbling and energy quantization)**.** *Let $u_k : \mathbb{S}^2 \to N$ be harmonic maps with uniformly bounded energy:*

$$E(u_k) \leq \Lambda.$$

Then, after passing to a subsequence, there exist:

- *a harmonic map $u_\infty : \mathbb{S}^2 \to N$,*
- *finitely many points $a_1, \dots, a_\ell \in \mathbb{S}^2$,*
- *for each j, finitely many nonconstant harmonic spheres*

$$\omega_{j,1}, \dots, \omega_{j,m_j} : \mathbb{S}^2 \to N,$$

such that:

(1) *(Smooth convergence away from bubbles)*

$$u_k \to u_\infty \quad \text{in } C^\infty_{\text{loc}}\big(\mathbb{S}^2 \setminus \{a_1, \dots, a_\ell\}\big).$$

(2) *(Energy identity)*

$$\lim_{k \to \infty} E(u_k) = E(u_\infty) + \sum_{j=1}^{\ell} \sum_{\alpha=1}^{m_j} E(\omega_{j,\alpha}).$$

(3) *(Energy quantization) Each bubble satisfies*

$$E(\omega_{j,\alpha}) \geq E_{\min}(N).$$

Hence

$$\ell \leq \frac{\Lambda}{E_{\min}(N)},$$

so only finitely many bubbles can occur.

References

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