LECTURE 5: ε - REGULARITY THEOREM AND PARTIAL REGULARITY FOR HARMONIC MAPS

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Let $u : \Omega \subset \mathbb{R}^n \to (N, h)$ be a map into a compact Riemannian manifold N (isometrically embedded in $\mathbb{R}^k$). The *Dirichlet energy* is

$$E(u; \Omega) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx,$$

and its critical points under compactly supported variations are the *harmonic maps.* In extrinsic form (with $N \hookrightarrow \mathbb{R}^k$), the Euler–Lagrange equation becomes

$$\Delta u = A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of $N \subset \mathbb{R}^k$.

Let $u : \Omega \subset \mathbb{R}^n \to N$ be a smooth harmonic map, and set

$$e := |\nabla u|^2.$$

Then the Bochner identity for harmonic maps gives

$$\Delta e = 2|\nabla^2 u|^2 - 2\big\langle R^N(\nabla u, \nabla u)\nabla u, \nabla u\big\rangle, \tag{0.1}$$

where R^N denotes the Riemann curvature tensor of the target manifold N.

Since N is compact, its curvature is bounded, and hence

$$-\Delta e \leq C\, e^2 \qquad \text{in } \Omega, \tag{0.2}$$

for a constant $C > 0$ depending only on the geometry of N. The inequality (0.2) is understood in the weak (distributional) sense for $u \in W^{1,2}$.

1. Epsilon-regularity

Theorem 1.1 (Epsilon-regularity in dimension 2)**.** *There exists $\varepsilon_0 > 0$ such that the following holds. If $u : B_1 \subset \mathbb{R}^2 \to N$ is a weak harmonic map satisfying*

$$\int_{B_1} |\nabla u|^2 \leq \varepsilon_0,$$

then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C \left(\int_{B_1} |\nabla u|^2 \right)^{1/2},$$

where C depends only on the geometry of N. In particular, $u \in C^\infty(B_{1/2})$.

Proof. We want to apply Moser iteration to (0.1)

Fix $0<\rho<R\le 1$, and let $\eta\in C_c^\infty(B_R)$ satisfy

$$\eta\equiv 1 \text{ on } B_\rho, \qquad 0\le\eta\le 1, \qquad |\nabla\eta|\le\frac{2}{R-\rho}.$$

For $p\ge 1$, multiply (0.1) by η^2e^{p-1} and integrate by parts, we get

$$\begin{aligned}
\int\Delta e\eta^2e^{p-1}&\le\int C\eta^2e^{p+1}\\
\int 2\eta e^{p-1}\langle\nabla e,\nabla\eta\rangle+\int(p-1)\eta^2e^{p-2}|\nabla e|^2&\le\int C\eta^2e^{p+1}\\
-2\int|\nabla e||\nabla\eta|\eta e^{p-1}+\int(p-1)\eta^2e^{p-2}|\nabla e|^2&\le\int C\eta^2e^{p+1}\\
-\int\frac{2}{p-1}|\nabla\eta|^2e^p-\int\frac{p-1}{2}\eta^2e^{p-2}|\nabla e|^2+\int(p-1)\eta^2e^{p-2}|\nabla e|^2&\le\int C\eta^2e^{p+1}
\end{aligned}$$

Namely

$$\int|\nabla(\eta e^{p/2})|^2\le C\int|\nabla\eta|^2e^p+C\int\eta^2e^{p+1}. \tag{1.1}$$

By Sobolev embedding $W^{1,2}\mapsto L^q$ compactly for $q=\frac{2n}{n-2}$, we have

$$\|e\|_{L^{\frac{p}{2}\cdot\frac{2n-\delta}{n-2}}(B_r)}\le C_\eta\|e\|_{L^{p+1}(B_R)}.$$

In dimension $n=2$, this directly implies L^q bound of u for any $q>1$

In dimension $n\ge 3$, by choosing $\delta<\frac{4p+4-2n}{p}$ we get an improvement as

$$\frac{p}{2}\cdot\frac{2n-\delta}{n-2}>p+1.$$

Moser iteration then implies L^∞ bound of e by L^{p+1} bound of e. (Although the energy bound only implies L^1 norm of e, so we don't a priori have L^{p+1} bound of u to get L^∞ by the Dirichlet energy.)

□

This can be generalized to the case of minimizing harmonic maps in higher dimensions.

Theorem 1.2 (Schoen–Uhlenbeck ε–regularity)**.** *Let $n\ge 3$ and let N be a compact Riemannian manifold. There exist constants $\varepsilon_0>0$ and $C>0$, depending only on n and the geometry of N, such that the following holds.*

If $u\in W^{1,2}(B_r(x_0),N)$ is an energy-minimizing harmonic map *and*

$$r^{2-n}\int_{B_r(x_0)}|\nabla u|^2\le\varepsilon_0, \tag{1.2}$$

then u is smooth in $B_{r/2}(x_0)$ and satisfies the estimate

$$\sup_{B_{r/2}(x_0)} |\nabla u| \leq C\, r^{-1} \left(r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \right)^{1/2}. \tag{1.3}$$

2. Estimate on the size of singular set and partial regularity

Using the epsilon regularity result, we say that the energy minimizing harmonic maps are regular away from a measure 0 set. Indeed, one can estimate the Hausdorff dimension of the singular set.

Theorem 2.1 (Schoen–Uhlenbeck partial regularity)**.** *Let $n \geq 3$ and let N be a compact Riemannian manifold. Suppose that*

$$u \in W^{1,2}(\Omega, N)$$

is an energy-minimizing harmonic map. *Then there exists a closed set*

$$\Sigma \subset \Omega$$

such that

(1) $u \in C^\infty(\Omega \setminus \Sigma)$;
(2) *the singular set Σ has Hausdorff dimension at most $n-2$, i.e.*

$$\dim_{\mathcal{H}}(\Sigma) \leq n - 2.$$

Proof. An important tool we use is the notion of energy density and the monotonicity formula.

For $x \in \Omega$ and $r > 0$ with $B_r(x) \subset \Omega$, define the scaled energy

$$\theta(x, r) := r^{2-n} \int_{B_r(x)} |\nabla u|^2.$$

Since u is an energy-minimizing (hence stationary) harmonic map, the monotonicity formula implies that $\theta(x, r)$ is nondecreasing in r. Therefore the limit

$$\theta(x, 0^+) := \lim_{r \downarrow 0} \theta(x, r)$$

exists for every $x \in \Omega$.

Let $\varepsilon_0 > 0$ be the constant from the ε–regularity theorem and define the singular set

$$\Sigma := \left\{ x \in \Omega : \theta(x, 0^+) \geq \varepsilon_0 \right\}.$$

By the ε–regularity theorem, u is smooth in $B_{r_0/2}(x_0)$. Hence

$$u \in C^\infty(\Omega \setminus \Sigma).$$

It is not hard to see that the regular set is open and the singular set is closed.

Now fix a compact set $\Omega' \Subset \Omega$. For each $x \in \Sigma \cap \Omega'$, by definition of Σ and monotonicity, there exists $r_x > 0$ such that $B_{5r_x}(x) \subset \Omega$ and

$$\int_{B_{r_x}(x)} |\nabla u|^2 \geq \varepsilon_0\, r_x^{n-2}. \tag{2.1}$$

The family $\{B_{r_x}(x)\}_{x\in\Sigma\cap\Omega'}$ is a covering of $\Sigma\cap\Omega'$. By the Vitali covering lemma, there exists a countable subcollection $\{B_{r_i}(x_i)\}$ such that

(1) the balls $B_{r_i}(x_i)$ are pairwise disjoint,

(2)
$$\Sigma\cap\Omega'\subset\bigcup_i B_{5r_i}(x_i).$$

So by (2.1) and the disjointness of the balls,

$$\varepsilon_0\sum_i r_i^{n-2}\le\sum_i\int_{B_{r_i}(x_i)}|\nabla u|^2=\int_{\cup_i B_{r_i}(x_i)}|\nabla u|^2\le\int_\Omega|\nabla u|^2<\infty.$$

Therefore
$$\sum_i r_i^{n-2}\le\frac{1}{\varepsilon_0}\int_\Omega|\nabla u|^2.$$

Using the covering by $B_{5r_i}(x_i)$ and the definition of Hausdorff measure, we conclude

$$\mathcal{H}^{n-2}(\Sigma\cap\Omega')\le C(n)\sum_i(5r_i)^{n-2}\le\frac{C(n)}{\varepsilon_0}\int_\Omega|\nabla u|^2<\infty.$$

Since $\Omega'\Subset\Omega$ was arbitrary, it follows that $\mathcal{H}^{n-2}(\Sigma)$ is locally finite in Ω, and hence
$$\dim_{\mathcal{H}}(\Sigma)\le n-2.$$

□

References

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