LECTURE 4: DE GOIRGI NASH-MOSER THEORY

SHENGWEN WANG

The De Giorgi-Nash-Moser theory in elliptic PDE is providing the initial L^∞ and $C^{0,\alpha}$ regularity, before higher regularity theory like Schauder theory applies. We will present here the Moser's approach for getting the local boundedness (which is iteration on the exponents of L^p norms, while De Giorgi's approach is iteration on the super-level sets).

1. Initial L^∞ bound and Moser iteration

Let

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$$

be an elliptic operator of divergence form and $U \subset \mathbb{R}^n$ a bounded open domain as before. Suppose the coefficients satisfies

$$\|a^{ij}\|_{L^\infty(U)}, \|c\|_{L^q(U)} \leq L_0$$

and

$$\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi^2|, \lambda, \Lambda > 0. \tag{1.1}$$

Our first theorem is the local boundedness of u, which only requires L^p boundedness of the coefficients and inhomogeneous term.

We will present a simplified case with $c = 0$ for the homogeneous equation, the general case follows by exactly the same argument with more technicality involved in absorbing the extra terms.

Theorem 1.1 ($L^p \to L^\infty$ estimate)**.** *Suppose $u \in H^1(B_1)$ is a subsolution, i.e.*

$$\int_{B_1} a^{ij} D_i u D_j \phi \leq 0, \quad \forall \phi \in H_0^1(B_1), \phi \geq 0. \tag{1.2}$$

Then for any $\theta \in (0,1)$, we have in the smaller ball B_θ that

$$\sup_{B_\theta} u^+ \leq C \frac{\|u^+\|_{L^p(B_1)}}{(1-\theta)^{\frac{n}{p}}},$$

for some positive constant $C = C(n.\lambda, \Lambda, p, q)$.

Proof. For $k > 0$ to be determined and $m \in \mathbb{N}$, we define

$$\bar{u} = u^+ + k,$$

and

$$\bar{u}_m = \begin{cases} \bar{u}, & u < m \\ k+m, & u \geq m \end{cases} .$$

Then one observes

$$k \le \bar{u}_m \le m + k,$$

and

$$\bar{u}_m \equiv \text{Constant}, D\bar{u}_m = 0, \quad \text{when } u < 0 \text{ or } u > m.$$

We choose the following non-netative test function

$$0 \le \phi = \zeta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1),$$

for some $\beta \ge 0, \zeta \in C_0^1(B_1)$.

We compute

$$\begin{aligned} D\phi =& 2\zeta D\zeta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + \zeta^2 \beta \bar{u}_m^{\beta-1} D\bar{u}_m \bar{u} + \zeta^2 \bar{u}_m^\beta D\bar{u} \\ =& 2\zeta D\zeta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + \zeta^2 \bar{u}_m^\beta (\beta D\bar{u}_m + D\bar{u}), \end{aligned}$$

where we used that $\bar{u}_m = \bar{u}$ when $D\bar{u}_m \ne 0$. Plugging into the equation (1.2) we get

$$\begin{aligned} 0 &\ge \int_{B_1} a^{ij} D_i u D_j \phi \\ &= \int_{B_1 \cap \{u>0\}} a^{ij} D_i u D_j \phi \\ &= \int_{B_1 \cap \{u>0\}} a^{ij} D_i u \cdot 2\zeta D\zeta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + a^{ij} D_i u \cdot \zeta^2 \bar{u}_m^\beta (\beta D\bar{u}_m + D\bar{u}) \\ &\ge \int_{B_1 \cap \{u>0\}} -\Lambda |D\bar{u}| \cdot 2\zeta |D\zeta| \cdot \bar{u}_m^\beta \bar{u} + \lambda\beta\zeta^2 |D\bar{u}_m|^2 \bar{u}_m^\beta + \lambda\zeta^2 |D\bar{u}|^2 \bar{u}_m^\beta \\ &\ge \int_{B_1 \cap \{u>0\}} [-\frac{1}{2}\lambda\zeta^2 |D\bar{u}|^2 \bar{u}_m^\beta - \frac{1}{2}\frac{\Lambda^2}{\lambda} \cdot 4|D\zeta|^2 \bar{u}_m^\beta \bar{u}^2] + \lambda\beta\zeta^2 |D\bar{u}_m|^2 \bar{u}_m^\beta + \lambda\zeta^2 |D\bar{u}|^2 \bar{u}_m^\beta \end{aligned} \tag{1.3}$$

(Here we used Cauchy-Schwarz for the first term in the previous line)

$$= \int_{B_1 \cap \{u>0\}} -2\frac{\Lambda^2}{\lambda} \cdot |D\zeta|^2 \bar{u}_m^\beta \bar{u}^2 + \lambda\beta\zeta^2 |D\bar{u}_m|^2 \bar{u}_m^\beta + \frac{1}{2}\lambda\zeta^2 |D\bar{u}|^2 \bar{u}_m^\beta.$$

Thus

$$\beta \int_{B_1} \zeta^2 |D\bar{u}_m|^2 \bar{u}_m^\beta + \int_{B_1} \zeta^2 |D\bar{u}|^2 \bar{u}_m^\beta \le C \int_{B_1} |D\zeta|^2 \bar{u}_m^\beta \bar{u}^2, \tag{1.4}$$

for some $C = C(\beta, \Lambda, \lambda)$.

(Notice that if $c \ne 0$ or there is an inhomogeneous term for the equation, then the LHS of (1.3) is not zero and one needs a few more steps in the absorption of terms.)

We see the RHS (1.4) is "roughly" an L^2 norm of the function

$$w = \bar{u}_m^{\frac{\beta}{2}} \bar{u},$$

whose L^2 norm of derivative is "roughly" bounded by the LHS of (1.4) as follows

$$|Dw|^2 = |\frac{\beta}{2} \bar{u}_m^{\frac{\beta}{2}-1} \bar{u} \nabla \bar{u}_m + \bar{u}_m^{\frac{\beta}{2}} \nabla \bar{u}|^2$$

$$
\begin{aligned}
&=|\frac{\beta}{2}\bar{u}_m^{\frac{\beta}{2}}\nabla\bar{u}_m+\bar{u}_m^{\frac{\beta}{2}}\nabla\bar{u}|^2\\
&\le(1+\beta)(\beta|D\bar{u}_m|^2\bar{u}_m^{\beta}+|D\bar{u}|^2\bar{u}_m^{\beta}).
\end{aligned}
$$

Namely (1.4) reads

$$\int_{B_1}|Dw|^2\zeta^2\le(1+\beta)\int_{B_1}|D\zeta|^2w^2.$$

By Sobolev inequality applied to the compactly supported ζw, we get

$$[\int_{B_1}|\zeta w|^{\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\le[\int_{B_1}|D(\zeta w)|^2]^{\frac12}\le\int_{B_1}|Dw|^2\zeta^2+\int_{B_1}|D\zeta|^2w^2]^{\frac12}\le[(2+\beta)\int_{B_1}|D\zeta|^2w^2]^{\frac12}.$$

Now we choose the cut-off function $\zeta\in C_0^1(B_1)$ so that for $0<r<R\le1$ there holds

$$
\begin{aligned}
&\zeta\equiv1,\quad\text{in }B_r\\
&0\le\zeta\le1,\quad\text{in }B_R\\
&|D\zeta|\le\frac{2}{R-r}.
\end{aligned}
$$

Noticing $\bar{u}_m\le\bar{u}$ and $\zeta\equiv1$ in B_r, we obtain

$$
\begin{aligned}
[\int_{B_r}\bar{u}_m^{(\beta+2)\frac{n}{n-2}}]^{\frac{n-2}{2n}}&=[\int_{B_r}\bar{u}_m^{\frac{\beta+2}{2}\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\\
&\le[\int_{B_r}(\bar{u}_m^{\frac{\beta}{2}}\bar{u})^{\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\\
&\le[\int_{B_r}w^{\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\\
&\le(\int_{B_1}|\zeta w|^{\frac{2n}{n-2}})^{\frac{n-2}{2n}}\\
&\le[(2+\beta)\int_{B_1}|D\zeta|^2w^2]^{\frac12}\\
&\le\frac{2\sqrt{2+\beta}}{(R-r)}[\int_{B_1}w^2]^{\frac12}\\
&\le\frac{2\sqrt{2+\beta}}{(R-r)}[\int_{B_1}\bar{u}^{\beta+2}]^{\frac12}.
\end{aligned}
$$

By letting $m\to\infty$, we can replace $\bar{u}_m$ by $\bar{u}$ and so

$$\|\bar{u}\|_{L^{p\chi}(B_1)}\le[\frac{2(p-1)}{(R-r)^2}]^{\frac1p}\|\bar{u}\|_{L^pB_R},$$

where $p=\beta+2,\chi=\frac{n}{n-2}$.

Here $p\chi > \gamma$ so we get an improvement from L^p to $L^{p\chi}$, as $p\chi > p$. After iterating i times with

$$p_i = 2\chi^i \to \infty, r_i = \theta + \frac{1}{2^i}(1-\theta) \to \theta, \quad i \in \mathbb{N},$$

we get

$$\begin{aligned}\|\bar{u}\|_{L^{p_i}(B_{r_i})} \leq & [\frac{4^i \cdot 2(p_i - 1)}{(1-\theta)^2}]^{\frac{1}{p_i}} \|\bar{u}\|_{L^{p_{i-1}}(r_{i-1})} \\ \leq & \Pi_{k=1}^{i} \left[4^i \cdot 2(p_i - 1)\right]^{\frac{1}{p_i}} \cdot \Pi_{k=1}^{i} \left[\frac{1}{(1-\theta)^2}\right]^{\frac{1}{p_i}} \cdot \|\bar{u}\|_{L^2(B_1)}.\end{aligned}$$

Taking the limit we get

$$\|\bar{u}\|_{L^\infty(B_\theta)} \leq C \cdot \frac{1}{(1-\theta)^2} \|\bar{u}\|_{L^2(B_1)}$$

□

2. Harnack inequality and Hölder regularity

As a consequence of Theorem 1.1, we also have Inf bound for non-negative super solutions.

Theorem 2.1. *Suppose $u \in H^1(B_1)$ is a non-negative supersolution, i.e.*

$$\int_{B_1} a^{ij} D_i u D_j \phi \geq 0, \quad \forall \phi \in H_0^1(B_1), \phi \geq 0. \tag{2.1}$$

Then for any $\theta \in (0,1), p \leq \frac{n}{n-2}$, we have in the smaller ball B_θ that

$$\inf_{B_\theta} u \geq C \|u\|_{L^p(B_1)},$$

for some positive constant $C = C(n.\lambda, \Lambda, p, q)$.

Idea of Proof: Apply Theorem 1.1 to $u^{-\beta}$ (so that a super-solution becomes a subsolution). The detail is left as an exercise. □

So combining Theorem 1.1 and Theorem 2.1, we have the Harnack inequality for actual non-negative solutions.

Theorem 2.2 (Moser-Harnack Inequality)**.** *Suppose $u \in H^1(B_1)$ is a non-negative weak solution, i.e.*

$$\int_{B_1} a^{ij} D_i u D_j \phi = 0, \quad \forall \phi \in H_0^1(B_R), \phi \geq 0. \tag{2.2}$$

We have

$$\sup_{B_R} u \leq C \inf_{B_R} u,$$

for some uniform constant $C = C(n, \lambda, \Lambda)$.

The Moser-Harnack inequality will then give us oscillation decay when the radius of balls shrinks, providing Hölder regularity (by standard iteration argument of Campanato).

Theorem 2.3 (Hölder continuity of weak solutions). *If $u \in H^1(U)$ is a weak solution to $Lu = 0$ in a bounded open domain $U \subset \mathbb{R}^n$, then for any $B_R(x_0) \Subset \Omega$ we have the following:*

- *For all $r \leq R$ we have*

$$\mathrm{osc}_{B_r(x_0)} u \leq C \left(\frac{r}{R}\right)^{\alpha} \mathrm{osc}_{B_R(x_0)} u,$$

for some $\alpha = \mu(n, \lambda, \Lambda) \in (0,1)$ and $C = C(n, \lambda, \Lambda) > 0$. Here

$$\mathrm{osc}_{B_\rho} u = \sup_{B_\rho} u - \inf_{B_\rho} u.$$

- *u is Hölder in a smaller ball, with the estimate*

$$R^{\alpha} \, [u]_{C^{0,\alpha}\left(B_{R/4}(x_0)\right)} \leq C \|u\|_{L^\infty(B_R)}.$$

Proof. First we prove the oscillation decay. Let

$$M := \sup_{B_R} u, \qquad m := \inf_{B_R} u, \qquad \omega := M - m.$$

Define

$$v := u - m \geq 0, \qquad w := M - u \geq 0 \quad \text{in } B_R.$$

Since the operator is linear and $Lu = 0$, we have $Lv = Lw = 0$.

By Moser's Harnack inequality, there exists $C_H > 1$ such that

$$\sup_{B_{R/2}} v \leq C_H \inf_{B_{R/2}} v, \qquad \sup_{B_{R/2}} w \leq C_H \inf_{B_{R/2}} w.$$

Set

$$A := \sup_{B_{R/2}} u, \qquad B := \inf_{B_{R/2}} u.$$

Applying Harnack to $v = u - m$ gives

$$A - m \leq C_H (B - m),$$

and applying it to $w = M - u$ gives

$$M - B \leq C_H (M - A).$$

Adding the above 2 bounds we get

$$(A - m) + (M - B) \leq C_H \big[(B - m) + (M - A)\big].$$

Using

$$(A - m) + (M - B) = \omega + (A - B), \qquad (B - m) + (M - A) = \omega - (A - B),$$

we obtain

$$\omega + (A - B) \leq C_H \big(\omega - (A - B)\big).$$

Rearranging,

$$(C_H + 1)(A - B) \leq (C_H - 1)\omega,$$

hence

$$\mathrm{osc}_{B_{R/2}} u = A - B \leq \frac{C_H - 1}{C_H + 1} \, \omega.$$

Setting $\sigma = \dfrac{C_H - 1}{C_H + 1} \in (0,1)$ completes the proof of first part on oscillation decay.

From the oscillation decay theorem, there exists $\sigma \in (0,1)$ such that

$$\operatorname{osc}_{B_{R/2}} u \le \sigma \ \operatorname{osc}_{B_R} u.$$

Iterating this estimate, we obtain for all $k \in \mathbb{N}$,

$$\operatorname{osc}_{B_{R/2^k}} u \le \sigma^k \ \operatorname{osc}_{B_R} u. \tag{1}$$

Choose $\alpha > 0$ such that

$$\sigma = 2^{-\alpha}, \quad \text{i.e.} \quad \alpha = -\frac{\log \sigma}{\log 2}.$$

Let $0 < \rho \le R$ and choose $k \in \mathbb{N}$ satisfying

$$\frac{R}{2^{k+1}} < \rho \le \frac{R}{2^k}.$$

Then, by monotonicity of oscillation in the radius,

$$\operatorname{osc}_{B_\rho} u \le \operatorname{osc}_{B_{R/2^k}} u \le \sigma^k \operatorname{osc}_{B_R} u = 2^{-\alpha k} \operatorname{osc}_{B_R} u.$$

Since $\rho \le R/2^k$, we have $2^{-k} \le \rho/R$, and hence

$$\operatorname{osc}_{B_\rho} u \le \left(\frac{\rho}{R}\right)^\alpha \operatorname{osc}_{B_R} u.$$

Finally, for any $x, y \in B_{R/2}$, setting $\rho = |x-y|$ yields

$$|u(x) - u(y)| \le \operatorname{osc}_{B_\rho} u \le C|x-y|^\alpha,$$

which proves $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$.

□