LECTURE 3: EXISTENCE THEORY FOR ELLIPTIC PDES

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We treat the existence theory and regularity theory separately in many PDE problems. First we need to obtain the existence of weak solution in some larger space, and then we proceed to prove the regularity / partial regularity of the solution. There are several approaches using abstract tools from functional analysis or topological arguments. We will present the Lax-Milgram theorem for existence of solutions to linear PDE and a fixed point argument that can be applied to non-linear problems.

1. Lax-Milgram and weak solutions

Consider an elliptic operator of divergence form defined before,

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u,$$

i.e. the coefficients satysfying

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \lambda > 0.$$

W define the bilinear form associated to L by

$$B_L[u,v] =: \int_U u \cdot Lv dx = \int_U -a^{ij}(x)u_{x_i}v_{x_j} + c(x)u(x)v(x)dx \tag{1.1}$$

for $u, v \in H_0^1(U), U \subset \mathbb{R}^n$.

Definition 1.1. A weak solution $u \in H^1(U)$ of the Dirichlet problem

$$\begin{aligned} Lu &= f \\ u|_{\partial U} &= 0, \end{aligned} \tag{1.2}$$

if

$$B_L[u,v] = \langle f, v\rangle_{L^2(U)}, \quad \forall v \in H_0^1(U). \tag{1.3}$$

The following functional analytic theorem will give existence of weak solutions.

Theorem 1.2 (Lax-Milgram)**.** *Let H be a real Hilbert space and B a bilinear form on H:*

$$B : H \times H \to \mathbb{R},$$

satisfying

$$|B[u,v]| \leq \alpha\|u\|\|v\|, \quad \forall u, v \in H, \tag{1.4}$$

$$\beta\|u\|^2 \leq B[u,u], \quad \forall u \in H, \tag{1.5}$$

for some $\alpha, \beta > 0$. Then for any $f \in H$, there exists a unique $u \in H$ so that

$$B[u,v] = \langle f, v\rangle.$$

Sketch of Proof: The existence of u is a consequence of Riesz Representation theorem for bounded (by (1.4)) linear functional, and the uniqueness part follows from the coercivity (1.4). □

Indeed, we can characterise exactly when the coercivity condition in the Lax-Milgram theorem fails.

Theorem 1.3 (Fredholm alternative)**.** *For an elliptic operator L satisfying the conditions stated at the beginning of the section, one of the following items hold:*

- *For each $f \in L^(U)$, there eixsts a unique weak solution u of the boundary value problem*

$$\begin{cases} Lu = f, & \textit{in } U \\ u = 0, & \textit{on } \partial U \end{cases}.$$

- *There exists a nontrivial weak solution $u \not\equiv 0$ to the homogenous problem*

$$\begin{cases} Lu = 0, & \textit{in } U \\ u = 0, & \textit{on } \partial U \end{cases}.$$

Indeed, when $c(x) \leq 0$, the first case always happens.

Corollary 1.4. *If the elliptic operator $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ satisfies $c(x) \leq 0$, then the first case in Fredholm alternative holds.*

Proof. The second case does not happen, because the weak maximum principle proved in Lecture 1 (for $c \leq 0$) guarantees the uniqueness of solutions. □

We can also deal with the case with non-zero boundary data.

Proposition 1.5. *Suppose L is an elliptic operator $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ satisfying $c(x) \leq 0$ and $U \subset \mathbb{R}^n$ is a smooth bounded domain. Then the following boundary value problem has a unique solution for any $f \in C^{0,\alpha}(U), \phi \in C^{2,\alpha}(U)$:*

$$\begin{cases} Lu = f, & \textit{in } U \\ u = \phi, & \textit{on } \partial U \end{cases}. \tag{1.6}$$

Proof. We can extend ϕ to a $\Phi \in C^{2,\alpha}(U)$ so that $\Phi|_{\partial U} = \phi$, as U is smooth.Then by the previous Corollary, there exists a unique solution for the following problem

$$\begin{cases} Lv = f - L\Psi, & \text{in } U \\ v = 0, & \text{on } \partial U \end{cases}.$$

Then we see that $f - L\Psi \in C^{0,\alpha}(U) \subset L^2(U)$ because the operator is second order, and thus $u = v + \Psi$ is a solution (1.6). □

Combining the existence of a weak solution (1.3) in H_0^1 for linear elliptic equations, we can apply the De Giorgi Nash Moser Moser theory in Lecture 4 to gain L^∞ and Hölder regularity, and then apply Schauder theory to get higher regularity for such solutions.

2. Leray-Schauder existence theory

The second existence theory is based on fixed point argument and can be applied to quasi-linear PDEs, e.g. minimal surface equation.

Recall the classical Brouwer's fixed point theorem, which states that "a continuous map from closed unit ball in $\mathbb{R}^n$ to itself must have have a fixed point", can be generalised to maps of compact convex sets of Banach spaces.

Theorem 2.1 (Schauder's fixed point theorem, Generlised Brouwer's)**.** *Let K be a compact convex set in a Banach space $\mathcal{B}$ and let $T : K \to K$ be continuous. Then T has a fixed point.*

As a corollary, one gets

Corollary 2.2. *Let $\mathcal{B}$ be a Banach space and $B \subset \mathcal{B}$ is its open unit ball. Suppose $T : \bar{B} \to \mathcal{B}$ is a continuous map such that*

- *The map T is compact, i.e. images of any compact set is precompact.*
- *$T(\partial B) \subset B$.*

Then T has a fixed point.

Sketch of Proof: Apply the Schauder's fixed point theorem to the map $T^\star : \bar{B} \to \bar{B}$ defined by

$$T^\star(x) = \begin{cases} T(x), & \text{for } \|T(x)\| \leq 1 \\ \frac{T(x)}{\|T(x)\|}, & \text{for } \|T(x)\| \geq 1 \end{cases},$$

and notice that the fixed point cannot happen at ∂B because $|T(y)| < 1 = |y|, \forall y \in \partial B$. ☐

The fixed point theorem can be

Theorem 2.3 (Leray-Schauder fixed point theorem)**.** *Let $\mathcal{B}$ be a Banach space and*

$$T : \mathcal{B} \times [0,1] \to \mathcal{B}$$

a compact map such that:

- *$T(x, 0) = 0$ for each $x \in \mathcal{B}$;*
- *There exists a constant $M > 0$ so that for each $(x,t) \in \mathcal{B} \times [0,1]$ which satisfies $x = T(x,t)$, there holds $\|x\| < M$.*

Then there is a fixed point $y \in \mathcal{B}$ of the map $T(\cdot, 1) : \mathcal{B} \to \mathcal{B}$ given by $T(y,1) = y$.

Proof. Without loss of generality, we may assume $M = 1$. Otherwise one can just rescale the norm on Banach space by a factor of $\frac{1}{M}$ and notice that a fixed point is unchanged by this norm scaling. For any $\varepsilon \in (0,1)$, we define a map from the closed unit ball,

$$T_\varepsilon : \bar{B} \to \mathcal{B}$$

$$T_\varepsilon(x) =: \begin{cases} T(\frac{x}{\|x\|}, \frac{1-\|x\|}{\varepsilon}), & \text{if } 1-\varepsilon \leq \|x\| \leq 1 \\ T(\frac{x}{1-\varepsilon}, 1), & \text{if } \|x\| \leq 1-\varepsilon \end{cases}.$$

For each ε, we see the image of ∂B by T_ε is

$$T_\varepsilon(\partial B) = T(\partial B, \frac{1-1}{\varepsilon}) = T(\partial B, 0) = 0,$$

by the definition of $T(\cdot, 0)$. So the Corollary 2.2 implies that there is a fixed point x_ε of T_ε for any ε. We define further that

$$t_\varepsilon =: \begin{cases} \frac{1-\|x_\varepsilon\|}{\varepsilon}, & \text{if } 1-\varepsilon \le \|x_\varepsilon\| \le 1 \\ 1, & \text{if } \|x_\varepsilon\| \le 1-\varepsilon \end{cases},$$

which is the second parameter of T_ε for this fixed point.

By compactness of T, we can find a subsequence so that

$$(x_{\varepsilon_k}, t_{\varepsilon_k}) \to (\hat{x}, \hat{t}) \in \bar{B} \times [0, 1].$$

There are 2 possible cases:

- If $t < 1$, then for ε_k small enough, there holds $t_{\varepsilon_k} < 1$, and thus
$$\|x_{\varepsilon_k}\| \ge 1 - \varepsilon_k \to 1 = \|x\|.$$
But this is a contradiction to the second condition that $\|x\| < M = 1$ as a fixed point $x = T(x, t)$.
- If $t = 1$, and thus $x = T(x, 1)$ gives a fixed point for $T(\cdot, 1)$ as desired.

□

We want to apply the Leray-Schauder theorem to the existence theory of quasi-linear elliptic PDEs of the form:

$$\partial_i(a^{ij}(x, u, \nabla u)\partial_j u) + c(x)u = 0, \quad \text{in } U \subset \mathbb{R}, \tag{2.1}$$

where the coefficients a^{ij}, c are $C^{0,\alpha}$ about every components of their variable and $c(x) \le 0$.

Theorem 2.4 (Quasi-linear existence)**.** *Let $\alpha \in (0, 1)$, $U \subset \mathbb{R}^n$ a bounded smooth open domain and $\phi \in C^{2,\alpha}(\bar{U})$. Suppose further that for some $\beta \in (0, 1)$, there exists $M > 0$ constant so that the following holds: For every $t \in [0, 1]$, each $C^{2,\alpha}$ solution u (not assuming it exists) of*

$$\begin{cases} \partial_i(a^{ij}(x, u, \nabla u)\partial_j u) + c(x)u = 0 & \text{in } U, c \le 0, \\ u = t\phi & \text{on } \partial U, \end{cases} \tag{2.2}$$

satisfies the a priori estimate

$$\|u\|_{C^{1,\beta}(\bar{U})} < M.$$

Then the Dirichlet problem

$$\begin{cases} \partial_i(a^{ij}(x, u, \nabla u)\partial_j u) + c(x)u = 0 & \text{in } U, \\ u = \phi & \text{on } \partial U \end{cases} \tag{2.3}$$

has a solution in $C^{2,\alpha}(\bar{U})$.

Proof. We define an operator

$$\begin{aligned} T :& C^{1,\beta}(\bar{U}) \times [0,1] \to C^{1,\beta}(\bar{U}) \\ T(v,t) =& u, \end{aligned}$$

where $u = T(v,t)$ is the unique solution of the linear problem obtained by replacing $u, \nabla u$ with $v, \nabla v$ in the coefficients a^{ij}.

$$\begin{cases} \partial_i(a^{ij}(x,v,\nabla v)\partial_j u) + tc(x)u = 0 \quad \text{in } U, \\ u = t\phi \quad \text{on } \partial U, \end{cases}$$

We see that any solution u of (2.2) is a fixed point of T. And by the assumption of the theorem, any such fixed point $u = T(u,t), t \in [0,1]$ must satisfies

$$\|v\|_{C^{1,\beta}(\bar{U})} < M.$$

So the Leray-Schauder fixed point theorem implies the existence of a fixed point for the map $T(\cdot, 1)$, namely a solution to (2.3).

The $C^{2,\alpha}$ regularity of the solution is coming from Schauder estimates, as the coefficients are now in $C^{0,\beta}$ (as u is in $C^{1,\beta}$). □

3. Example: Existence of solution to minimal surface equation

The area of graph of u over $U \subset \mathbb{R}^n$ is

$$A_u(U) = \int_U \sqrt{1 + |\nabla u|^2} dx.$$

For any compactly supported variation $\phi \in C_0^\infty(U)$ we have the first variation formula

$$\begin{aligned} 0 =& \frac{d}{dt}|_{t=0} A_{u+t\phi}(U) \\ =& \int_U \frac{d}{dt}\sqrt{1 + |\nabla u + t\nabla\phi|^2} dx|_{t=0} \\ =& \int_U \frac{1}{2\sqrt{1 + |\nabla u + t\nabla\phi|^2}} 2\langle \nabla u + t\nabla\phi, \nabla\phi\rangle dx|_{t=0} \\ =& \int_U \langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nabla\phi\rangle dx \\ =& -\int_U \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)\phi dx. \end{aligned}$$

So the Euler-Lagrange equation of the area functional is the minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{3.1}$$

To apply the Leray-Schauder estimate above, we need to prove a priori gradient estimates.

Lemma 3.1. *Let $U \subset \mathbb{R}^n$ be a mean convex bounded smooth domain and $\phi \in C^\infty(\bar{U})$. Then any solution to the boundary value minimal surface equation:*

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) =& 0 \quad \text{in } U \\ u =& \phi \quad \text{on } \partial U, \end{aligned} \tag{3.2}$$

satisfies the gradient estimate

$$\|\nabla u\|_{C^{0,\beta}(\bar{U})} < M,$$

for some $M > 0$ and $\beta \in (0,1)$.

The proof of this Hölder bounds of gradient will use De Goirgi - Nash - Moser theory, which is the material of next lecture.

As a application of Theorem 2.4 and Lemma 3.1, we have

Theorem 3.2. *Let $U \subset \mathbb{R}^n$ and ϕ satisfy the same conditions as in Lemma 3.1. There exists a unique smooth solution to* (3.2).

Remark 3.3. One gets $C^{2,\alpha}$ regularity by Leray-Schauder, then the higher regularity follows from Schauder estimates from Lecture 2.

Remark 3.4. This is actually a minimising solution (i.e. the surface with least area with that prescribed boundary). In general, minimising hypersurfaces with prescribe boundary are only smooth up to dimension 8 (and in higher dimension may have singular set of codimension 8). However, when they are graphical, we know that they are actually smooth in all dimensions.

4. Variational method

Another approach to the existence of minimal surface is through a variational method. Let u_k be a minimizing sequence of the area functional so that

$$A_{u_k}(U) \to \inf_{u \in C^\infty(U), u|_{\partial U} = \phi} A_u(U).$$

In order to get a subsequence converging to a limit, we need some estimates and compactness in function space.

Since the minimising sequence have uniformly bounded area, we have

$$\int_U |\nabla u_k| \leq \int_U \sqrt{1+|\nabla u_k|^2} \leq A_0.$$

And by Poincare inequality and that $u_k - \Phi = 0$ on ∂U (where Ψ is an extension of ϕ to the interior of U), we get

$$\int_U |u_k - \Psi| \leq \int_U |\nabla u_k| \leq A_0,$$

and so

$$\int_U |u_k| \leq C(A_0, \phi).$$

Thus the sequence is uniformly bounded in $W^{1,1}(U)$. Sobolev inequality gives that $W^{1,1} \mapsto L^{\frac{n}{n-1}}$ compactly. So the existence of a solution in L^p for $p \leq \frac{n}{n-1}$ follows by taking a subsequential limit

$$u_{k_k} \to u_0.$$

For higher regularity, we need to Apply De Goirgi Nash - Moser theory from next lecture and Schauder theory from the last lecture.