LECTURE 2: SCHAUDER ESTIMATES

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We will present the proof given by Simon [2] using blow-up argument and Liouville's (rigidity) theorem.

Denote by

$$[u]_{C^{0,\alpha}(U)} = \sup_{x,y\in U, x\neq y} \frac{u(x)-u(y)}{|x-y|^\alpha}$$

the Hölder semi-norm and

$$\|u\|_{C^{k,\alpha}} = \sum_{|\beta|\leq k} \|u\|_{L^\infty(U)} + [D^k]_{C^{0,\alpha}(U)}$$

the Hölder $C^{k,\alpha}$ norm.

First, let's prove the interior estimate for Laplace operator.

Proposition 0.1 (Interior estimate for Laplace). *Let $U \subset \mathbb{R}^n$ be a bounded open domain.If $u \in C^{2,\alpha}(U)$ satisfies*

$$\Delta u = f,$$

in an open domain $U \subset \mathbb{R}^n$, then for any pre-compact open subset $K \Subset U$ we have

$$[\mathrm{Hess}_u]_{C^{0,\alpha}(K)} \leq C[\Delta u]_{C^{0,\alpha}(U)} = C[f]_{C^{0,\alpha}(U)},$$

for some constant $C = C(n, \alpha, K, U)$.

Proof. We will reduce it to case in a disk by the following claim.

Claim 0.2. *It suffices to prove that*

$$[\mathrm{Hess}_u]_{C^{0,\alpha}(B_1)} \leq C[\Delta u]_{C^{0,\alpha}(B_R)} = C[f]_{C^{0,\alpha}(B_R)},$$

for some constant $C = C(n, \alpha)$ and large enough R.

Proof of Claim. We will leave this as an exercise (using covering argument by using disks contained in U and with centres in K). □

Indeed, by scaling, it suffices to prove the Claim with $r = 1$.

Suppose the claim does not hold, then there exists sequences $u_k \in C^{2,\alpha}(B_{R_k}), f_k \in C^{0,\alpha}(B_{R_k})$ and $R_k > k$,so that

$$[\mathrm{Hess}_{u_k}]_{C^{0,\alpha}(B_1)} > k[f_k]_{C^{0,\alpha}(B_{R_k})}, k = 1, 2, \ldots.$$

We can replace u_k, f_k by $\frac{u_k}{\|\mathrm{Hess}_{u_k}\|_{C^{0,\alpha}(B_1)}}, \frac{f_k}{\|\mathrm{Hess}_{u_k}\|_{C^{0,\alpha}(B_1)}}$ and assume without loss of generality that

$$[\mathrm{Hess}_{u_k}]_{C^{0,\alpha}(B_1)} = 1$$

$$[f_k]_{C^{0,\alpha}(B_{R_k})} < \frac{1}{k} \to 0.$$

By the definition of Hölder norm, there exists $x_k, y_k \in B_1$, so that

$$\frac{|D^2_{ij}u_k(y_k) - D^2_{ij}u_k(x_k)|}{|y_k - x_k|^\alpha} > c_n > 0,$$

for some dimensional constant c_n.

Since $R_k > k \to \infty$ and u_k, by Arzela-Ascoli, we can extract a subsequence (which we still index by k without loss of generality) so that

$$u_k \to u \quad \text{in } C^{2,\beta}, \beta < \alpha,$$
$$x_k \to x_\infty \in B_2.$$

Without loss of generality again, we can subtract the function u_k by a degree-2 polynomial (degree-2 polynomial has Hölder norms of Hessian being 0, thus not affecting these inequalities) so that the above still hold and moreover

$$\begin{aligned} u(x_k) =& 0, \\ \nabla u(x_k) =& 0, \\ \text{Hess}_u(x_k) =& 0. \end{aligned}$$

Combining these, we have that the limit satisfies

$$\begin{aligned} \Delta u =& 0, \\ |\text{Hess}_u|_\alpha \leq& 1, \\ u(x_\infty) =& 0, \\ \nabla u(x_\infty) =& 0, \\ \text{Hess}_u(x_\infty) =& 0, \end{aligned} \tag{0.1}$$

and one of the following holds:

Case 1: $y_k \to y_\infty \neq x_\infty$. In this case $\text{Hess}_u(y_\infty) > c_n|y_\infty - x_\infty|^\alpha \neq 0$.

Case 2: $y_k \to x_\infty$. In this case, we can rescale consider the rescaled sequence $\tilde{u}_k(x) = \frac{1}{|y_k - x_k|^{2+\alpha}} u_k(x_k + |y_k - x_k|x)$ defined in a ball of radius $\frac{R_k}{|y_k - x_k|}$ that also converges to an entire harmonic function satisfying (0.1) with 0 in place of x_∞ and that $\text{Hess}_u(\frac{y_\infty - x_\infty}{|y_\infty - x_\infty|}) \neq 0$.

In either case above, we get an entire harmonic function that has distinct Hessian at 2 different points and that

$$\sup_{B_r} |u| \leq Cr^{2+\alpha} \leq C^{3-\varepsilon}, \varepsilon = \frac{1-\alpha}{2}.$$

On the other hand, by the Liouville Theorem (Corollary **??**), u is a polynomial of degree at most 2, which is a contradiction to either case above that have non-constant Hessian! □

Next, we generalise the above interior Schauder estimates for Laplace operator to general elliptic operators.

Theorem 0.3 (Interior Schauder for elliptic operators). *Let $U \subset \mathbb{R}^n$ be a bounded open domain and we consider an elliptic operator as in (??) $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ with $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \forall x \in U, \xi \in \mathbb{R}^n$ and $a^{ij}, c \in C^{0,\alpha}(U)$. If $u \in C^{2,\alpha}(U)$ satisfies*

$$Lu = f,$$

in an open domain $U \subset \mathbb{R}^n$, then for any pre-compact open subset $K \Subset U$ we have

$$\|u\|_{C^{2,\alpha}(K)} \leq C(\|Lu\|_{C^{0,\alpha}(U)} + \|u\|_{L^\infty(U)}) = C(\|f\|_{C^{0,\alpha}(U)} + \|u\|_{L^\infty(U)}),$$

for some constant $C = C(n, \alpha, K, U)$.

Proof. Again it suffices to prove the estimate for disks. We choose $K = B_r$ and $U = B_{2r}$ and then use covering argument to extend to general K, U as in the previous proposition.

Choose a fixed $x_0 \in B_r$. Since the coefficietns a^{ij}, c are Hölder continuous, we have

$$a^{ij}(x_0)\partial_i\partial_j u = Lu - (a^{ij}(x) - a^{ij}(x_0))\partial_i\partial_j u - \partial_i a^{ij}(x)\partial_j u - c(x)u.$$

By ellipticity of a^{ij} and compactness of B_{2r} we get

$$\Lambda|\xi|^2 \geq a^{ij}(x_0)\xi_i\xi_j \geq \lambda|\xi|^2.$$

And thus by the previous Proposition 0.1 (applied with $K = B_r, U = B_{2r}$) we have

$$\begin{aligned}[\mathrm{Hess}_u]_{C^{0,\alpha}(B_r)} \leq& C[\Delta u]_{C^{0,\alpha}(B_{2r})} \\ \leq& C(\lambda, \Lambda)[a^{ij}(x_0)\partial_i\partial_j u]_{C^{0,\alpha}(B_{2r})} \\ \leq& C(\lambda, \Lambda, \|a^{ij}\|_{C^{0,\alpha}(B_{2r})}) \left([Lu]_{C^{0,\alpha}(B_{2r})} + r^\alpha[\mathrm{Hess}_u]_{C^{0,\alpha}(B_{2r})} + \|u\|_{C^2(B_{2r})}\right).\end{aligned}$$

By choosing $r < \frac{1}{2C(\lambda, \Lambda, \|a^{ij}\|_{C^{0,\alpha}(B_{2r})})}$ small enough we then have

$$[\mathrm{Hess}_u]_{C^{0,\alpha}(B_r)} \leq C\left([Lu]_{C^{0,\alpha}(B_{2r})} + \|u\|_{C^2(B_{2r})}\right). \tag{0.2}$$

Finally, we close the argument by applying the following interpolation inequality.

Lemma 0.4 (Interpolation inequality).

For any $\varepsilon \in (0,1)$ there exists C_ε so that the following holds for any $u \in C^2(B_{2\rho})$ and $\rho \in (0,1)$:

$$\rho^2\|u\|_{C^2(B_\rho)} \leq \varepsilon\rho^{2+\alpha}\|u\|_{C^{2,\alpha}(B_{2\rho})} + C_\varepsilon\|u\|_{L^\infty(B_{2\rho})}. \tag{0.3}$$

Proof of Lemma. This is also proved by compactness (contradiction argument) using Arzela-Ascoli. See Problem set 2. □

We denote by

$$Q := \sup_{x \in B_2} \mathrm{dist}(x, \partial B_2)^2|\mathrm{Hess}_u(x)|,$$

and notice that the supremum is attained in the interior ($\mathrm{dist}(x, \partial B_2)^2|D^2u(x)| = 0 \quad$ on ∂B_2) for some $x_0 \in B_2$. Let $\rho = \frac{1}{3}\mathrm{dist}(x_0, \partial B_2)$, we get $B_{2\rho}(x_0) \subset B_2$ and

$$\begin{aligned}Q =& 9\rho^2|\mathrm{Hess}_u(x_0)| \\ \leq& 9\rho^2\|\mathrm{Hess}_u\|_{L^\infty(B_\rho(x_0))}\end{aligned}$$

$$
\begin{aligned}
&\le 9\varepsilon\rho^{2+\alpha}\|u\|_{C^{2,\alpha}(B_{2\rho}(x_0))} + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))} \quad \text{by (0.3)}\\
&\le 9\varepsilon\rho^{2+\alpha}\left([\mathrm{Hess}_u]_{C^{0,\alpha}(B_{2\rho}(x_0))} + \|u\|_{C^2(B_{2\rho}(x_0))}\right) + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))}\\
&\le 9\varepsilon\rho^{2+\alpha}\left[C\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + C\|u\|_{C^2(B_{2\rho}(x_0))}\right] + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))} \quad \text{by (0.2)}\\
&\le 9\varepsilon\rho^{2+\alpha}\left[C\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + C[\mathrm{Hess}_u]_{C^{0,\alpha}(B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))}\\
&\le (9\varepsilon C + 9C_\varepsilon)\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C\varepsilon\rho^2[\mathrm{Hess}_u]_{L^\infty(B_{2\rho}(x_0))}\\
&\le \tilde{C}(\varepsilon, C)\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C\varepsilon \sup_{x\in B_{2\rho}(x_0))} \mathrm{dist}(x,\partial B_2)^2[\mathrm{Hess}_u]_{L^\infty(B_{2\rho}(x_0))}\\
&\le \tilde{C}\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C\varepsilon Q.
\end{aligned}
$$

By choosing $\varepsilon < \frac{1}{18C}$ and absorbing the second term on the right hand side to the left, we get

$$
Q \le \tilde{C}\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] \le \tilde{C}[\|Lu\|_{C^{0,\alpha}(B_2)} + \|u\|_{L^\infty(B_2)}].
$$

Applying (0.2) on compact subsets of bounded open sets (could easily see by covering argument), we get

$$
\begin{aligned}
|u\|_{C^{2,\alpha}(B_1)} =& |u\|_{C^2(B_1)} + [\mathrm{Hess}_u]_{C^{0,\alpha}(B_1)}\\
\le& |u\|_{C^2(B_1)} + C[\|Lu\|_{C^{0,\alpha}(B_{\frac{3}{2}})} + \|u\|_{C^2(B_{\frac{3}{2}})}]\\
\le& C_1[\|Lu\|_{C^{0,\alpha}(B_{\frac{3}{2}})} + \|u\|_{L^\infty(B_{\frac{3}{2}})} + 2(\frac{3}{2})^2[\mathrm{Hess}_u]_{L^\infty(B_{\frac{3}{2}r}(x_0))}]\\
\le& C_2[\|Lu\|_{C^{0,\alpha}(B_{\frac{3}{2}})} + \|u\|_{L^\infty(B_{\frac{3}{2}})} + Q]\\
\le& \tilde{C}[\|Lu\|_{C^{0,\alpha}(B_2)} + \|u\|_{L^\infty(B_2)}].
\end{aligned}
$$

□

References

[1] Qing Han and Fanghua Lin, Elliptic Partial Differential Equations, Second Edition, *Courant Lecture Notes in Mathematics, American Mathematical Society, 2011.*

[2] Leon Simon, Schauder estimates by scaling, *Calculus of Variations and Partial Differential Equations, Volume 5, pages 391-407, 1997*