LECTURE 1: HARMONIC FUNCTIONS AND A PRIORI ESTIMATES

SHENGWEN WANG

Let $u \in C^2(U)$ be a solution to the Laplace equation

$$\Delta u = 0, \quad \text{on a domain } U \subset \mathbb{R}^n. \tag{0.1}$$

The existence of solutions for more general classess of elliptic PDEs could follow several approaches, see Lecture 3 for examples.

In modern PDE theory, we usually separate existence and regularity part. First we use the weak formulation of (0.1)

$$\int_U u\Delta\eta = 0, \forall \eta \in C_c^\infty(U), \tag{0.2}$$

to get existence of solution in some Sobolev space, say $H^1_{loc}(U)$. And then we prove the regularity / partial regularity of the solutions.

Indeed for harmonic functions, one can improve the regularity of solutions to C^∞, and even analytic! (See also Hilbert's XIX problem.)

We also have quantitative bounds on all its derivatives. Let's start with some a priori estimates. Although there are now various ways to see the regularity of harmonic functions, but these estimates still give models for studying more general elliptic PDEs.

1. Some a priori estimates

The first is an interior estimate.

Theorem 1.1 (Gradient bound)**.** *If u is a harmonic function on $B_r(x_0) \subset R^n$ above, then we have*

$$|\nabla u(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B_{x_0}(r))} \leq \frac{C}{r^{\frac{n+2}{2}}} \|u\|_{L^2(B_{x_0}(r))}, \tag{1.1}$$

for a dimensional constant $C = C(n)$.

Proof. By the mean value inequality, we have

$$\begin{aligned} |U_{x_i}(x_0)| &= \left| \frac{2^n}{\omega_n r^n} \int_{B_{\frac{r}{2}(x_0)}} u_{x_i} dx \right| \\ &= \left| \frac{2^n}{\omega_n r^n} \int_{B_{\frac{r}{2}(x_0)}} \operatorname{div}(u\mathbf{e_i}) dx \right| \\ &= \left| \frac{2^n}{\omega_n r^n} \int_{\partial B_{\frac{r}{2}(x_0)}} \langle u\mathbf{e_i}, \mathbf{n} \rangle dS \right| \end{aligned}$$

$$
\begin{aligned}
&\le \left| \frac{2^n}{\omega_n r^n} \omega_{n-1} \left(\frac{r}{2}\right)^{n-1} \|u\|_{L^\infty(\partial B_{\frac{r}{2}}(x_0))} \right| \\
&\le \left| \frac{2^n}{\omega_n r^n} \omega_{n-1} \left(\frac{r}{2}\right)^{n-1} \frac{2^n}{\omega_n r^n} \|u\|_{L^1(B_r(x_0))} \right| \\
&= \frac{C(n)}{r^{n+1}} \|u\|_{L^1(B_r(x_0))},
\end{aligned}
$$

where the second last line used $\partial B_{\frac{r}{2}}(x_0) \subset B_r(x_0)$ and the mean value inequality.

□

In a similar way, one can obtain interior bounds on all higher derivatives by the L^1 norm, this will be left as an exercise.

Exercise 1.2. Under the same assumption as in Theorem 1.1, we have

$$D^\alpha u(x_0) \le \frac{C}{r^{n+k}} \|u\|_{L^1(B_{x_0}(r))},$$

with the norm of multiindex α being $|\alpha| = k$.

As a consequence of this gradient estimates, we have the Liouville Theorem for entire harmonic function.

Corollary 1.3 (Liouville's Theorem)**.** *Let $C < \infty, \varepsilon > 0$. If $u : \mathbb{R}^n \to \mathbb{R}$ is a harmonic function with $\sup_{B_r(0)} |u| \le C r^{k-\varepsilon}$ for some positive integer $k \in \mathbb{N}$, then u must be a polynomial of degree at most $k-1$.*

In particular, entire harmonic functions with sub-linear growth must be constant.

Proof. The proof is an easy exercise from the derivative estimate above. □

Remark 1.4. This will be used later to give a version of proof of Schauder estimate for higher regularity.

Next, we provide a global Hölder regularity estimate up to the boundary.

Theorem 1.5. *Let u be a harmonic function on $\overline{B_1(0)}$ with $u|_{\partial B_1(0)} = \phi$. Suppose $\phi \in C^\alpha(\partial B_1(0))$ for some $\alpha \in (0,1)$, then we have*

$$\|u\|_{C^{\frac{\alpha}{2}}(\overline{B_1(0)})} \le C \|\phi\|_{C^\alpha(\partial B_1(0))}, \tag{1.2}$$

for some constant $C = C(n, \alpha)$.

Proof. Without loss of generality, up to translation, we assume $x_0 = (1, 0, \dots, 0)$ and $\phi(0) = 0$. For $x \in \partial B_1(x_0)$, we then have

$$|x^2| = 2x_1.$$

We denote by

$$K =: \sup_{x \in \partial B_1(x_0)} \frac{|\phi(x)|}{|x|^\alpha},$$

and

$$v(x) =: 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}}.$$

Then

$$\begin{aligned}\Delta v(x) =& 2^{\frac{\alpha}{2}} K \cdot \frac{\alpha}{2}(\frac{\alpha}{2}-1)x_1^{\frac{\alpha}{2}-2} < 0 = \Delta u, \\ v|_{\partial B_1(x_0)} =& 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}} = K|x|^\alpha \geq \phi(x) = u|_{\partial B_1(x_0)}.\end{aligned}$$

Applying maximum principle to the subharmonic function $u - v$ we then get

$$|u(x)| \leq |v(x)| =^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}}, \forall x \in B_1(x_0),$$

namely

$$\sup_{x \in B_1(x_0)} \frac{|u(x) - u(0)|}{|x-0|^{\frac{\alpha}{2}}} \leq 2^{\frac{\alpha}{2}} \sup_{x \in \partial B_1(x_0)} \frac{|\phi(x) - \phi(0)|}{|x-0|^2}. \tag{1.3}$$

Here 0 can be replaced by arbitrary point on the boundary $\partial B_1(x_0)$ by translation and rotation.

We will leave the rest of the proof as an exercise. □

Exercise 1.6. Prove that (1.3) implies (1.2).

2. Maximum principle and barriers

The maximum principle is a key tool in elliptic PDEs that plays an important role in a priori estimates and regularity theory.

We consider operators of divergence form (they arise from variational problems natually)

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u, \tag{2.1}$$

It is said to be elliptic in $U \subset \mathbb{R}^n$ if there exists a $\lambda > 0$ so that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \forall x \in U, \xi \in \mathbb{R}^n.$$

We further assume that $a^{ij}, c \in C^{0,\alpha}(U)$ are Hölder continuous.

The maximum principles are true for operators of non-divergence form too. But here by making use of the divergence form and variational structure, we have a simple proof of weak maximum principle.

Theorem 2.1 (Weak Maximum Principle)**.** *Let L be an elliptic operator as above and the coefficient $c(x) \leq 0$. If*

$$\begin{aligned}Lu \geq& 0 \quad \text{in } U \\ u|_{\partial U} \leq& 0,\end{aligned}$$

then we have

$$u \leq 0, \quad \text{in } U.$$

Proof. We use the test function

$$u^+ = \max\{u, 0\}.$$

Using integration by parts and ellipticity, we get

$$\begin{aligned}
\int_U [\partial_i(a^{ij}(x)\partial_j u^+ + c(x)u]u^+ dx \geq& 0\\
\int_U [\partial_i(a^{ij}(x)\partial_j u^+ + c(x)u^+]u^+ dx \geq& 0\\
\int_U -a^{ij}(x)\partial_j u^+ \partial_i u^+ + c(x)(u^+)^2]dx \geq& 0\\
\int_U -\lambda|\nabla u^+|^2 + c(x)(u^+)^2]dx \geq& 0, \qquad (2.2)
\end{aligned}$$

which forces $u^+ = 0$ (namely $u \leq 0$) because $c \leq 0$ and $\lambda > 0$. □

Remark 2.2. Notice that the proof works for weak solutions $u \in H^1$.

Moreover, if the volume of region U is small, we can remove the assumption on negativity of c.

Theorem 2.3 (Small Volume Maximum Principle)**.** *Let L be an elliptic operator as above. There exists a $\delta > 0$ so that the following hold: If*

$$\begin{aligned}
Lu \geq& 0 \quad \textit{in } U\\
u|_{\partial U} \leq& 0\\
|U| <& \delta,
\end{aligned}$$

then we have

$$u \leq 0, \quad \textit{in } U.$$

Proof. By Faber-Krahn, the first eigenvalue of the region U satisfies

$$\lambda_1(U) \geq \lambda_1(B_R) = \frac{c_n}{R^2} > 0,$$

where B_R is a round disk of radius R so that its volume $|B| = |U|$. Namely

$$\int_U |\nabla u^+|^2 \geq \lambda_1(U) \int_U \lambda|u^+|^2 \geq \frac{c_n}{R^2} \int_U \lambda|u^+|^2.$$

Plugging into (2.2) we get

$$\int_U -\frac{\lambda c_n}{R^2}|u^+|^2 + c(x)(u^+)^2]dx \geq 0.$$

This again forces $u^+ = 0$ (namely $u \leq 0$) when the volume $|\Omega| = \omega_n R^n$ is small enough (namely R is small enough). □

Remark 2.4. The conclusion is in general not true if c does not have a sign and the volume of domain is not small. For example $u(x) = \sin x$ in $(0, 2\pi)$ achieves both maximum and minimum in the interior.

At the points that u achieves maximum on the boundary, we have the following properties on outer derivative

Theorem 2.5 (Hopf Lemma)**.** *Let L be an elliptic operator as above and the coefficient $c(x) \leq 0$. If*

$$\begin{aligned} Lu \geq & 0 \quad \text{in } U, \\ u|_U < & \max_{\partial U} u \quad \text{in the interior of } U, \end{aligned}$$

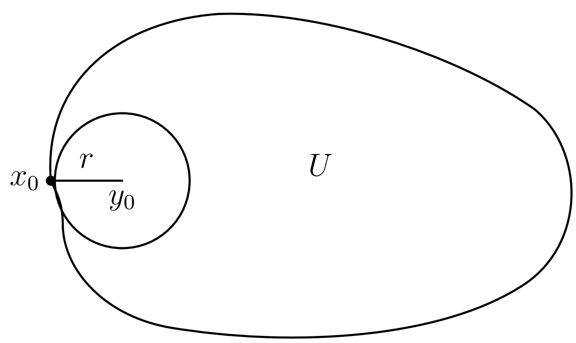
and $x_0 \in \partial U$ be a point that achieves non-negative maximum

$$u(x_0) = \max_{\partial U} u \geq 0,$$

such that x_0 satisfies the interior ball condition (namely there is a ball $B_r(y_0) \subset U$ so that $x_0 \in \partial B_r(y_0)$), then we have

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the unit outer normal at x_0.



Proof.

Consider a barrier function

$$v(x) = u(x) - u(x_0) + \varepsilon[e^{-\alpha|x-y_0|^2} - e^{-\alpha r^2}] \quad \text{in } B_r(y_0) \setminus B_{\frac{r}{2}}(y_0).$$

We then have

$$\begin{aligned} Lv =& Lu - cu(x_0) + \varepsilon L e^{-\alpha|x-y_0|^2} \\ \geq & 0 + \varepsilon e^{-\alpha|x-y_0|^2}[4\alpha^2 a^{ij}(x)(x_i - y_{0,i})(x_j - y_{0,j}) - 2\sum_{i=1}^n \alpha a^{ii}(x) + \alpha \partial_j a^{ij}(x)(x_i - y_{0,i}) + c(x)] \\ \geq & \varepsilon e^{-\alpha|x-y_0|^2}[4\alpha^2 \lambda |x - y_0|^2 - 2\sum_{i=1}^n \alpha a^{ii}(x) + \alpha \partial_j a^{ij}(x)(x_i - y_{0,i})] \\ \geq & \varepsilon e^{-\alpha|x-y_0|^2}[\alpha^2 \lambda r^2 - 2\sum_{i=1}^n \alpha a^{ii}(x) + \alpha \partial_j a^{ij}(x)(x_i - y_{0,i})]. \end{aligned}$$

By choosing $\alpha > 0$ large enough, we get

(2.3) $$Lv > 0 \quad \text{in } B_r(y_0) \setminus B_{\frac{r}{2}}(y_0).$$

Moreover, on the boundary of the annulus $B_r(y_0) \setminus B_{\frac{r}{2}}(y_0)$ we get

$$\begin{aligned} v|_{B_r(y_0)} =& u(x) - u(x_0) \leq 0 \\ v|_{B_{\frac{r}{2}}(y_0)} \leq& \max_{\partial B_{\frac{r}{2}}(y_0)} u(x) - u(x_0) + \varepsilon C(\alpha, \lambda, r) < 0, \quad \text{for } \varepsilon \text{ chosen small enough.} \end{aligned} \tag{2.4}$$

Here in the second bound, we used that $u < u(x_0)$ in the interior of U and that $\partial B_{\frac{r}{2}}(y_0)$ is compact.

Combining (2.3) and (2.4) we get by the weak maximum principle that

$$v \leq 0 = v(x_0) \quad \text{in } U.$$

As a consequence, by continuity of v, we know

$$\begin{aligned} \frac{\partial v}{\partial \nu}(x_0) \geq& 0 \\ \frac{\partial u}{\partial \nu}(x_0) + \varepsilon[-2\alpha r e^{-\alpha r^2}] \geq& 0 \\ \frac{\partial u}{\partial \nu}(x_0) \geq& 2\varepsilon \alpha r e^{-\alpha r^2} > 0. \end{aligned}$$

□

As a consequence we have the strong maximum principle

Theorem 2.6 (Strong Maximum Principle)**.** *Let L be an elliptic operator as above and the coefficient $c(x) \leq 0$. If*

$$Lu \geq 0 \quad \textit{in } U,$$

then

$$u|_U < \max_{\partial U} u = \max_{\overline{U}} u \quad \textit{in the interior of } U,$$

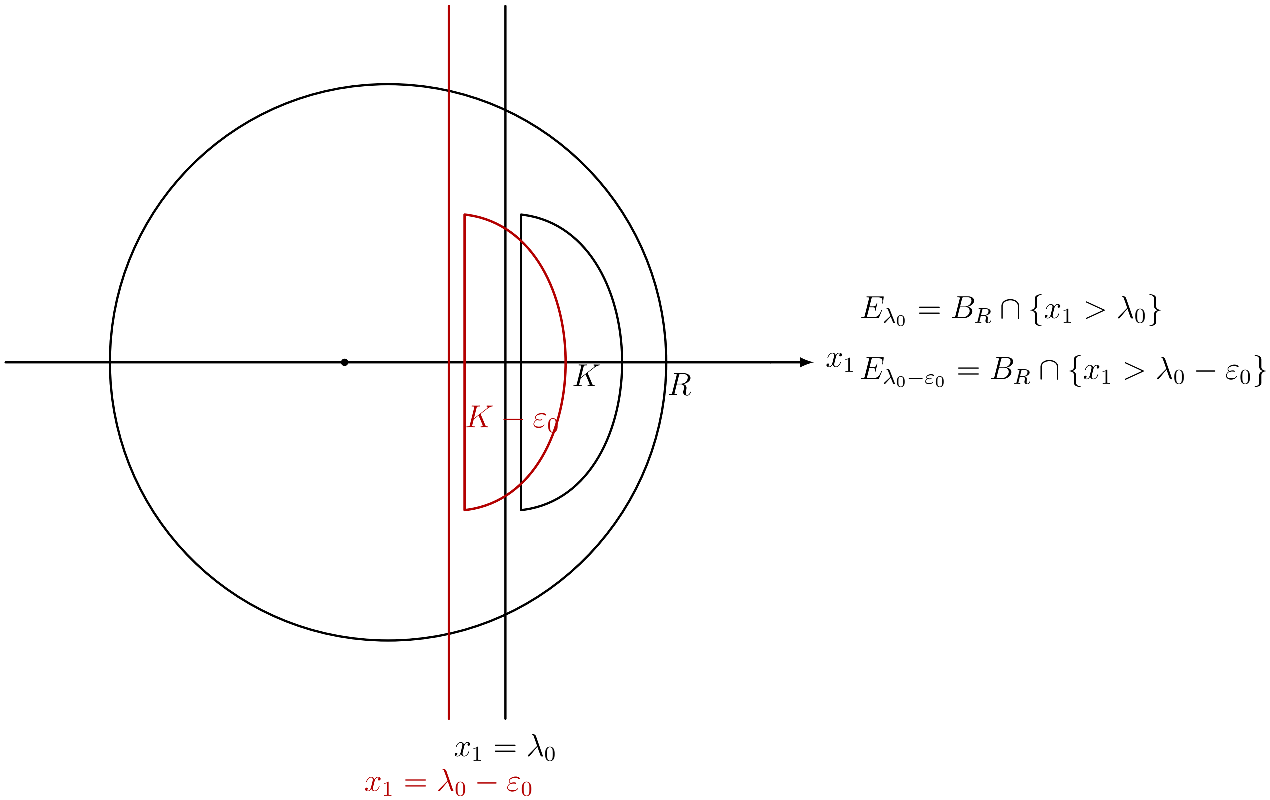
unless u is a constant.

The maximum principle has important geometric applications on proving uniqueness / rigidity / symmetry of solutions.

Theorem 2.7 (The moving plane method)**.** *If $u \in C^2(B_R)$ satisfies*

$$\begin{aligned} \Delta u =& f(u), \\ u >& 0, \quad \textit{in } B_R \\ u =& 0. \quad \textit{on } \partial B_R, \end{aligned}$$

then u is rotational symmetric, i,e. $u(x) = u(xe^{i\theta}), \forall \theta \in \mathbb{R}$.



Proof.
We will prove that u is reflexive symmetric about any hyperplanes passing through the origin, which implies rotational symmetry. Without loss of generality (up to rotation), we only need to prove symmetric about the plane $x_1 = 0$.

For any $\lambda \in [0, R]$, define

$$E_\lambda = B_R \cap \{x_1 > \lambda\},$$

and the function

$$v_\lambda = u(x) - u(2\lambda e_1 - x) \quad \text{on } E_\lambda.$$

Notice that for $\lambda > R - \varepsilon$ close enough to R, we have $|E_\lambda| < \delta$ (so that the small volume maximum principle is applicable). And

$$\begin{aligned} v_\lambda|_{\partial B_R \cap \{x_1 \geq \lambda\}}(x) \leq& 0 - u(2\lambda e_1 - x) < 0 \\ v_\lambda|_{B_R \cap \{x_1 = \lambda\}} =& 0. \end{aligned}$$

So $v_\lambda|_{\partial E_\lambda} \leq 0$ as $\partial E_\lambda = [\partial B_R \cap \{x_1 \geq \lambda\}] \cup [B_R \cap \{x_1 = \lambda\}]$. Thus the small volume maximum principle combined with strong maximum principle gives $v_\lambda|_{E_\lambda} < 0$ for $\lambda > R - \varepsilon$ when ε small enough.

We denote by

$$\lambda_0 = \inf_{v_\lambda \leq 0, \forall \lambda > \bar{\lambda}} \bar{\lambda}.$$

The proof will then be finished by the following claim:

Claim 2.8.

$$\lambda_0 = 0.$$

Proof of Claim. Suppose not, then $\lambda_0 > 0$. We then choose a compact subset $K \subset E_{\lambda_0}$ so that $|E_{\lambda_0} \setminus K| < \frac{\delta}{2}$. Since K is compact and $v_\lambda|_K < 0$, we can choose a $c_0 > 0$ so that $v_\lambda|_K < -c_0 < 0$. Next, we choose ε_0 small enough so that

$$|E_{\lambda_0 - \varepsilon_0} - (K - \varepsilon e_1)| < \delta,$$

and

$$v_\lambda|_{K - \varepsilon e_1} < -\frac{c_0}{2} < 0.$$

The small volume maximum principle applied to the region $E_{\lambda_0 - \varepsilon_0} - (K - \varepsilon e_1)$ then implies

$$v_\lambda|_{E_{\lambda_0 - \varepsilon_0}} \leq 0,$$

which is a contradiction to the definition of λ_0. ☐

With this claim in hand, we see that $v_0 \leq 0$ with $\lambda = 0$. And by reflection, we see that $v_0 \geq 0$ too. So $v_0 \equiv 0$ and thus u is reflexive symmetric about the planes $\{x_1 = 0\}$. Since the choice of such planes is arbitrary, we see the rotational symmetry in the disk B_R.

☐