ELLIPTIC PDES

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Contents

Introduction and outline of the course

This is the lecture notes for LTCC course "Elliptic Partial Differential Equations (Advanced)" taught in fall Spring 2026. The course contains five 2-hour lectures. We will focus primarily on divergence-form elliptic PDEs, which arise naturally from variational problems and the variational structure could simplified some of the technical proofs. The plan for the course is as follows:

In lecture 1, we gather some key regularity estimates for harmonic functions, which are extendable to more general elliptic PDEs.

In lecture 2, we will present the Schauder estimates, which gives higher order Hölder regularity of solutions.

In lecture 3, we will give some approaches to the existence theory of elliptic PDEs. In modern PDE, the existence and regularity theory are usually treated separately. One first obtain existence of weak solutions and then prove regularity of the solutions.

In lecture 4, we will go over the De Giorgi Nash-Moser theory, which provides the initial L^∞ and Hölder regularity of the solutions before applying Schauder estimates.

In lecture 5, we will talk about some applications in the geometric PDE of harmonic maps, e.g. the ε - regularity theorem and partial regularity theory.

1. Harmonic functions

Let $u \in C^2(U)$ be a solution to the Laplace equation

$$\Delta u = 0, \quad \text{on a domain } U \subset \mathbb{R}^n. \tag{1.1}$$

The existence of solutions for more general classess of elliptic PDEs could follow several approaches, see Chapter 2 for examples.

In modern PDE theory, we usually separate existence and regularity part. First we use the weak formulation of (1.1)

$$\int_U u\Delta\eta = 0, \forall \eta \in C_c^\infty(U), \tag{1.2}$$

to get existence of solution in some Sobolev space, say $H^1_{loc}(U)$. And then we prove the regularity / partial regularity of the solutions.

Indeed for harmonic functions, one can improve the regularity of solutions to C^∞, and even analytic! (See also Hilbert's XIX problem.)

We also have quantitative bounds on all its derivatives. Let's start with some a priori estimates. Although there are now various ways to see the regularity of harmonic functions, but these estimates still give models for studying more general elliptic PDEs.

1.1. **Some a priori estimates.**

The first is an interior estimate.

Theorem 1.1 (Gradient bound). *If u is a harmonic function on $B_r(x_0) \subset R^n$ above, then we have*

$$|\nabla u(x_0)| \le \frac{C}{r^{n+1}} \|u\|_{L^1(B_{x_0}(r))} \le \frac{C}{r^{\frac{n+2}{2}}} \|u\|_{L^2(B_{x_0}(r))}, \tag{1.3}$$

for a dimensional constant $C = C(n)$.

Proof. By the mean value inequality, we have

$$\begin{aligned}
|U_{x_i}(x_0)| &= \left| \frac{2^n}{\omega_n r^n} \int_{B_{\frac{r}{2}(x_0)}} u_{x_i} dx \right| \\
&= \left| \frac{2^n}{\omega_n r^n} \int_{B_{\frac{r}{2}(x_0)}} \operatorname{div}(u\mathbf{e_i}) dx \right| \\
&= \left| \frac{2^n}{\omega_n r^n} \int_{\partial B_{\frac{r}{2}(x_0)}} \langle u\mathbf{e_i}, \mathbf{n}\rangle dS \right| \\
&\le \left| \frac{2^n}{\omega_n r^n} \omega_{n-1} \left(\frac{r}{2}\right)^{n-1} \|u\|_{L^\infty(\partial B_{\frac{r}{2}(x_0)})} \right| \\
&\le \left| \frac{2^n}{\omega_n r^n} \omega_{n-1} \left(\frac{r}{2}\right)^{n-1} \frac{2^n}{\omega_n r^n} \|u\|_{L^1(B_r(x_0))} \right| \\
&= \frac{C(n)}{r^{n+1}} \|u\|_{L^1(B_r(x_0))},
\end{aligned}$$

where the second last line used $\partial B_{\frac{r}{2}}(x_0) \subset B_r(x_0)$ and the mean value inequality.

□

In a similar way, one can obtain interior bounds on all higher derivatives by the L^1 norm, this will be left as an exercise.

Exercise 1.2. Under the same assumption as in Theorem 1.1, we have

$$D^\alpha u(x_0) \leq \frac{C}{r^{n+k}} \|u\|_{L^1(B_{x_0}(r))},$$

with the norm of multiindex α being $|\alpha| = k$.

As a consequence of this gradient estimates, we have the Liouville Theorem for entire harmonic function.

Corollary 1.3 (Liouville's Theorem)**.** *Let $C < \infty, \varepsilon > 0$. If $u : \mathbb{R}^n \to \mathbb{R}$ is a harmonic function with $\sup_{B_r(0)} |u| \leq Cr^{k-\varepsilon}$ for some positive integer $k \in \mathbb{N}$, then u must be a polynomial of degree at most $k-1$.*

In particular, entire harmonic functions with sub-linear growth must be constant.

Proof. The proof is an easy exercise from the derivative estimate above. □

Remark 1.4. This will be used later to give a version of proof of Schauder estimate for higher regularity.

Next, we provide a global Hölder regularity estimate up to the boundary.

Theorem 1.5. *Let u be a harmonic function on $\overline{B_1(0)}$ with $u|_{\partial B_1(0)} = \phi$. Suppose $\phi \in C^\alpha(\partial B_1(0))$ for some $\alpha \in (0,1)$, then we have*

$$\|u\|_{C^{\frac{\alpha}{2}}(\overline{B_1(0)})} \leq C\|\phi\|_{C^\alpha(\partial B_1(0))}, \tag{1.4}$$

for some constant $C = C(n, \alpha)$.

Proof. Without loss of generality, up to translation, we assume $x_0 = (1, 0, \ldots, 0)$ and $\phi(0) = 0$. For $x \in \partial B_1(x_0)$, we then have

$$|x^2| = 2x_1.$$

We denote by

$$K =: \sup_{x \in \partial B_1(x_0)} \frac{|\phi(x)|}{|x|^\alpha},$$

and

$$v(x) =: 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}}.$$

Then

$$\begin{aligned} \Delta v(x) =& 2^{\frac{\alpha}{2}} K \cdot \frac{\alpha}{2}(\frac{\alpha}{2} - 1) x_1^{\frac{\alpha}{2}-2} < 0 = \Delta u, \\ v|_{\partial B_1(x_0)} =& 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}} = K|x|^\alpha \geq \phi(x) = u|_{\partial B_1(x_0)}. \end{aligned}$$

Applying maximum principle to the subharmonic function $u - v$ we then get

$$|u(x)| \leq |v(x)| =^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}}, \forall x \in B_1(x_0),$$

namely

$$\sup_{x \in B_1(x_0)} \frac{|u(x) - u(0)|}{|x - 0|^{\frac{\alpha}{2}}} \leq 2^{\frac{\alpha}{2}} \sup_{x \in \partial B_1(x_0)} \frac{|\phi(x) - \phi(0)|}{|x - 0|^2}. \tag{1.5}$$

Here 0 can be replaced by arbitrary point on the boundary $\partial B_1(x_0)$ by translation and rotation.

We will leave the rest of the proof as an exercise. □

Exercise 1.6. Prove that (1.5) implies (1.4).

1.2. **Maximum principle and barriers.** The maximum principle is a key tool in elliptic PDEs that plays an important role in a priori estimates and regularity theory.

We consider operators of divergence form (they arise from variational problems natually)

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u, \tag{1.6}$$

It is said to be elliptic in $U \subset \mathbb{R}^n$ if there exists a $\lambda > 0$ so that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \forall x \in U, \xi \in \mathbb{R}^n.$$

We further assume that $a^{ij}, c \in C^{0,\alpha}(U)$ are Hölder continuous.

The maximum principles are true for operators of non-divergence form too. But here by making use of the divergence form and variational structure, we have a simple proof of weak maximum principle.

Theorem 1.7 (Weak Maximum Principle)**.** *Let L be an elliptic operator as above and the coefficient $c(x) \leq 0$. If*

$$\begin{aligned} Lu \geq& 0 \quad \text{in } U \\ u|_{\partial U} \leq& 0, \end{aligned}$$

then we have

$$u \leq 0, \quad \text{in } U.$$

Proof. We use the test function

$$u^+ = \max\{u, 0\}.$$

Using integration by parts and ellipticity, we get

$$\begin{aligned} \int_U [\partial_i(a^{ij}(x)\partial_j u^+ + c(x)u]u^+ dx \geq& 0 \\ \int_U [\partial_i(a^{ij}(x)\partial_j u^+ + c(x)u^+]u^+ dx \geq& 0 \\ \int_U -a^{ij}(x)\partial_j u^+ \partial_i u^+ + c(x)(u^+)^2]dx \geq& 0 \end{aligned}$$

$$\int_U -\lambda|\nabla u^+|^2 + c(x)(u^+)^2]dx \geq 0, \tag{1.7}$$

which forces $u^+ = 0$ (namely $u \leq 0$) because $c \leq 0$ and $\lambda > 0$. □

Remark 1.8. Notice that the proof works for weak solutions $u \in H^1$.

Moreover, if the volume of region U is small, we can remove the assumption on negativity of c.

Theorem 1.9 (Small Volume Maximum Principle)**.** *Let L be an elliptic operator as above. There exists a $\delta > 0$ so that the following hold: If*

$$\begin{aligned} Lu \geq& 0 \quad \text{in } U \\ u|_{\partial U} \leq& 0 \\ |U| <& \delta, \end{aligned}$$

then we have

$$u \leq 0, \quad \text{in } U.$$

Proof. By Faber-Krahn, the first eigenvalue of the region U satisfies

$$\lambda_1(U) \geq \lambda_1(B_R) = \frac{c_n}{R^2} > 0,$$

where B_R is a round disk of radius R so that its volume $|B| = |U|$. Namely

$$\int_U |\nabla u^+|^2 \geq \lambda_1(U) \int_U \lambda |u^+|^2 \geq \frac{c_n}{R^2} \int_U \lambda |u^+|^2.$$

Plugging into (1.7) we get

$$\int_U -\frac{\lambda c_n}{R^2}|u^+|^2 + c(x)(u^+)^2]dx \geq 0.$$

This again forces $u^+ = 0$ (namely $u \leq 0$) when the volume $|\Omega| = \omega_n R^n$ is small enough (namely R is small enough). □

Remark 1.10. The conclusion is in general not true if c does not have a sign and the volume of domain is not small. For example $u(x) = \sin x$ in $(0, 2\pi)$ achieves both maximum and minimum in the interior.

At the points that u achieves maximum on the boundary, we have the following properties on outer derivative

Theorem 1.11 (Hopf Lemma)**.** *Let L be an elliptic operator as above and the coefficient $c(x) \leq 0$. If*

$$\begin{aligned} Lu \geq& 0 \quad \text{in } U, \\ u|_U <& \max_{\partial U} u \quad \text{in the interior of } U, \end{aligned}$$

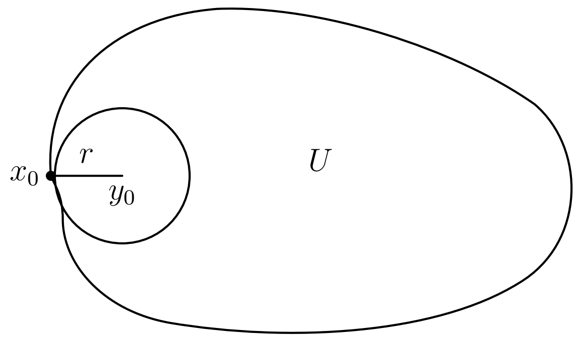
and $x_0 \in \partial U$ *be a point that achieves non-negative maximum*

$$u(x_0) = \max_{\partial U} u \geq 0,$$

such that x_0 *satisfies the interior ball condition (namely there is a ball* $B_r(y_0) \subset U$ *so that* $x_0 \in \partial B_r(y_0)$*), then we have*

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν *is the unit outer normal at* x_0*.*



Proof.

Consider a barrier function

$$v(x) = u(x) - u(x_0) + \varepsilon[e^{-\alpha|x-y_0|^2} - e^{-\alpha r^2}] \quad \text{in } B_r(y_0) \setminus B_{\frac{r}{2}}(y_0).$$

We then have

$$\begin{aligned} Lv =& Lu - cu(x_0) + \varepsilon L e^{-\alpha|x-y_0|^2} \\ \geq& 0 + \varepsilon e^{-\alpha|x-y_0|^2}[4\alpha^2 a^{ij}(x)(x_i - y_{0,i})(x_j - y_{0,j}) - 2\sum_{i=1}^n \alpha a^{ii}(x) + \alpha \partial_j a^{ij}(x)(x_i - y_{0,i}) + c(x)] \\ & \text{(Here we used } c < 0, u(x_0) \geq 0) \text{ so that } -cu(x_0) \geq 0.) \\ \geq& \varepsilon e^{-\alpha|x-y_0|^2}[4\alpha^2 \lambda |x-y_0|^2 - 2\sum_{i=1}^n \alpha a^{ii}(x) + \alpha \partial_j a^{ij}(x)(x_i - y_{0,i})] \\ \geq& \varepsilon e^{-\alpha|x-y_0|^2}[\alpha^2 \lambda r^2 - 2\sum_{i=1}^n \alpha a^{ii}(x) + \alpha \partial_j a^{ij}(x)(x_i - y_{0,i})]. \end{aligned}$$

By choosing $\alpha > 0$ large enough, we get

$$Lv > 0 \quad \text{in } B_r(y_0) \setminus B_{\frac{r}{2}}(y_0). \tag{1.8}$$

Moreover, on the boundary of the annulus $B_r(y_0) \setminus B_{\frac{r}{2}}(y_0)$ we get

$$\begin{aligned} v|_{B_r(y_0)} =& u(x) - u(x_0) \leq 0 \\ v|_{B_{\frac{r}{2}}(y_0)} \leq& \max_{\partial B_{\frac{r}{2}}(y_0)} u(x) - u(x_0) + \varepsilon C(\alpha, \lambda, r) < 0, \quad \text{for } \varepsilon \text{ chosen small enough.} \end{aligned} \tag{1.9}$$

Here in the second bound, we used that $u < u(x_0)$ in the interior of U and that $\partial B_{\frac{r}{2}}(y_0)$ is compact.

Combining (1.8) and (1.9) we get by the weak maximum principle that

$$v \le 0 = v(x_0) \quad \text{in } U.$$

As a consequence, by continuity of v, we know

$$\begin{aligned} \frac{\partial v}{\partial \nu}(x_0) \ge& 0 \\ \frac{\partial u}{\partial \nu}(x_0) + \varepsilon[-2\alpha r e^{-\alpha r^2}] \ge& 0 \\ \frac{\partial u}{\partial \nu}(x_0) \ge& 2\varepsilon \alpha r e^{-\alpha r^2} > 0. \end{aligned}$$

□

As a consequence we have the strong maximum principle

Theorem 1.12 (Strong Maximum Principle)**.** *Let L be an elliptic operator as above and the coefficient $c(x) \le 0$. If*

$$Lu \ge 0 \quad \textit{in } U,$$

then

$$u|_U < \max_{\partial U} u = \max_{\overline{U}} u \quad \textit{in the interior of } U,$$

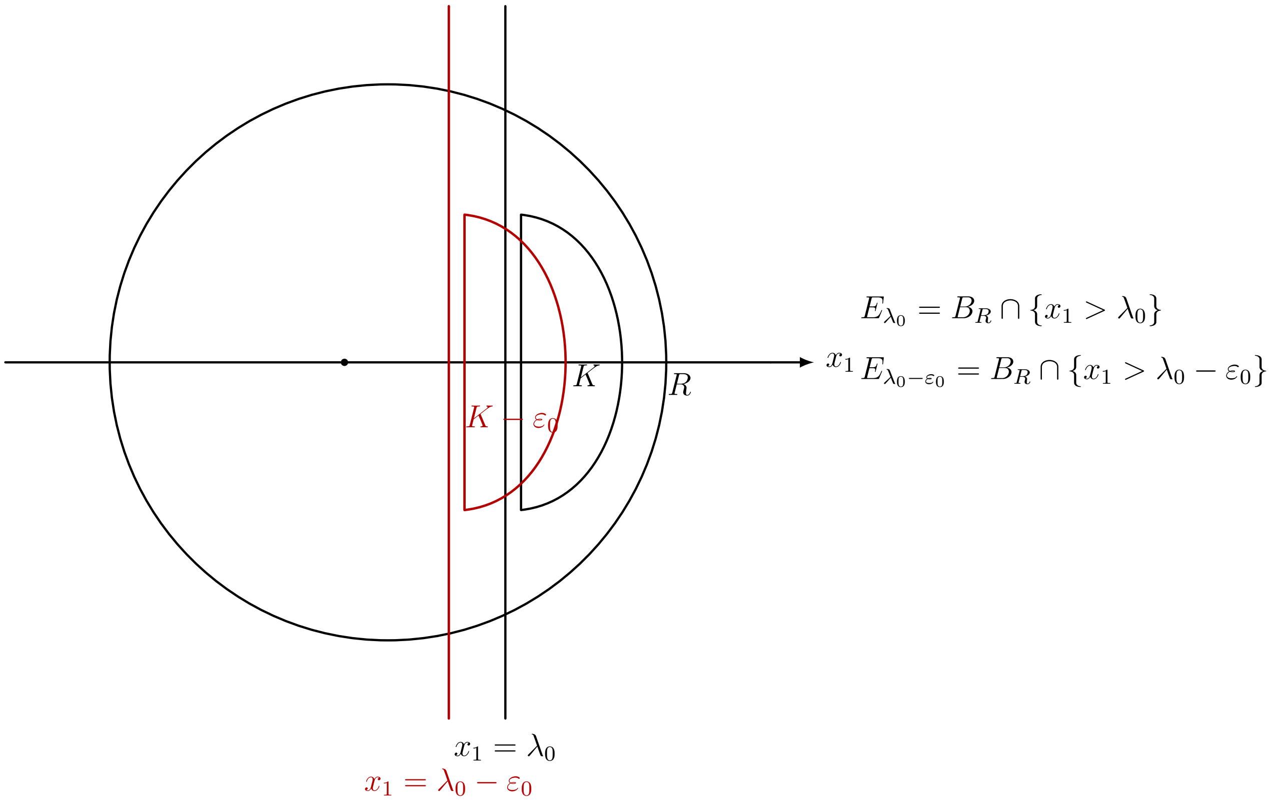
unless u is a constant.

The maximum principle has important geometric applications on proving uniqueness / rigidity / symmetry of solutions.

Theorem 1.13 (The moving plane method)**.** *If $u \in C^2(B_R)$ satisfies*

$$\begin{aligned} \Delta u =& f(u), \\ u >& 0, \quad \textit{in } B_R \\ u =& 0. \quad \textit{on } \partial B_R, \end{aligned}$$

then u is rotational symmetric, i,e. $u(x) = u(xe^{i\theta}), \forall \theta \in \mathbb{R}$.



Proof.

We will prove that u is reflexive symmetric about any hyperplanes passing through the origin, which implies rotational symmetry. Without loss of generality (up to rotation), we only need to prove symmetric about the plane $x_1 = 0$.

For any $\lambda \in [0, R]$, define

$$E_\lambda = B_R \cap \{x_1 > \lambda\},$$

and the function

$$v_\lambda = u(x) - u(2\lambda e_1 - x) \quad \text{on } E_\lambda.$$

Notice that for $\lambda > R - \varepsilon$ close enough to R, we have $|E_\lambda| < \delta$ (so that the small volume maximum principle is applicable). And

$$\begin{aligned} v_\lambda|_{\partial B_R \cap \{x_1 \geq \lambda\}}(x) \leq& 0 - u(2\lambda e_1 - x) < 0 \\ v_\lambda|_{B_R \cap \{x_1 = \lambda\}} =& 0. \end{aligned}$$

So $v_\lambda|_{\partial E_\lambda} \leq 0$ as $\partial E_\lambda = [\partial B_R \cap \{x_1 \geq \lambda\}] \cup [B_R \cap \{x_1 = \lambda\}]$. Thus the small volume maximum principle combined with strong maximum principle gives $v_\lambda|_{E_\lambda} < 0$ for $\lambda > R - \varepsilon$ when ε small enough.

We denote by

$$\lambda_0 = \inf_{v_\lambda \leq 0, \forall \lambda > \bar{\lambda}} \bar{\lambda}.$$

The proof will then be finished by the following claim:

Claim 1.14.

$$\lambda_0 = 0.$$

Proof of Claim. Suppose not, then $\lambda_0 > 0$. We then choose a compact subset $K \subset E_{\lambda_0}$ so that $|E_{\lambda_0} \setminus K| < \frac{\delta}{2}$. Since K is compact and $v_\lambda|_K < 0$, we can choose a $c_0 > 0$ so that $v_\lambda|_K < -c_0 < 0$. Next, we choose ε_0 small enough so that

$$|E_{\lambda_0-\varepsilon_0} - (K - \varepsilon e_1)| < \delta,$$

and

$$v_\lambda|_{K-\varepsilon e_1} < -\frac{c_0}{2} < 0.$$

The small volume maximum principle applied to the region $E_{\lambda_0-\varepsilon_0} - (K - \varepsilon e_1)$ then implies

$$v_\lambda|_{E_{\lambda_0-\varepsilon_0}} \leq 0,$$

which is a contradiction to the definition of λ_0. □

With this claim in hand, we see that $v_0 \leq 0$ with $\lambda = 0$. And by reflection, we see that $v_0 \geq 0$ too. So $v_0 \equiv 0$ and thus u is reflexive symmetric about the planes $\{x_1 = 0\}$. Since the choice of such planes is arbitrary, we see the rotational symmetry in the disk B_R.

□

2. Schauder estimates and higher regularity

We will present the proof given by Simon [2] using blow-up argument and Liouville's (rigidity) theorem.

Denote by

$$[u]_{C^{0,\alpha}(U)} = \sup_{x,y\in U, x\neq y} \frac{u(x)-u(y)}{|x-y|^\alpha}$$

the Hölder semi-norm and

$$\|u\|_{C^{k,\alpha}} = \sum_{|\beta|\le k} \|u\|_{L^\infty(U)} + [D^k]_{C^{0,\alpha}(U)}$$

the Hölder $C^{k,\alpha}$ norm.

First, let's prove the interior estimate for Laplace operator.

Proposition 2.1 (Interior estimate for Laplace)**.** *Let $U \subset \mathbb{R}^n$ be a bounded open domain.If $u \in C^{2,\alpha}(U)$ satisfies*

$$\Delta u = f,$$

in an open domain $U \subset \mathbb{R}^n$, then for any pre-compact open subset $K \Subset U$ we have

$$[\mathrm{Hess}_u]_{C^{0,\alpha}(K)} \le C[\Delta u]_{C^{0,\alpha}(U)} = C[f]_{C^{0,\alpha}(U)},$$

for some constant $C = C(n, \alpha, K, U)$.

Proof. We will reduce it to case in a disk by the following claim.

Claim 2.2. *It suffices to prove that*

$$[\mathrm{Hess}_u]_{C^{0,\alpha}(B_1)} \le C[\Delta u]_{C^{0,\alpha}(B_R)} = C[f]_{C^{0,\alpha}(B_R)},$$

for some constant $C = C(n, \alpha)$ and large enough R.

Proof of Claim. We will leave this as an exercise (using covering argument by using disks contained in U and with centres in K). □

Indeed, by scaling, it suffices to prove the Claim with $r = 1$.

Suppose the claim does not hold, then there exists sequences $u_k \in C^{2,\alpha}(B_{R_k}), f_k \in C^{0,\alpha}(B_{R_k})$ and $R_k > k$,so that

$$[\mathrm{Hess}_{u_k}]_{C^{0,\alpha}(B_1)} > k[f_k]_{C^{0,\alpha}(B_{R_k})}, k = 1, 2, \dots.$$

We can replace u_k, f_k by $\frac{u_k}{\|\mathrm{Hess}_{u_k}\|_{C^{0,\alpha}(B_1)}}, \frac{f_k}{\|\mathrm{Hess}_{u_k}\|_{C^{0,\alpha}(B_1)}}$ and assume without loss of generality that

$$[\mathrm{Hess}_{u_k}]_{C^{0,\alpha}(B_1)} = 1$$
$$[f_k]_{C^{0,\alpha}(B_{R_k})} < \frac{1}{k} \to 0.$$

By the definition of Hölder norm, there exists $x_k, y_k \in B_1$, so that

$$\frac{|D^2_{ij}u_k(y_k) - D^2_{ij}u_k(x_k)|}{|y_k - x_k|^\alpha} > c_n > 0,$$

for some dimensional constant c_n.

Since $R_k > k \to \infty$ and u_k, by Arzela-Ascoli, we can extract a subsequence (which we still index by k without loss of generality) so that

$$u_k \to u \quad \text{in } C^{2,\beta}, \beta < \alpha,$$
$$x_k \to x_\infty \in B_2.$$

Without loss of generality again, we can subtract the function u_k by a degree-2 polynomial (degree-2 polynomial has Hölder norms of Hessian being 0, thus not affecting these inequalities) so that the above still hold and moreover

$$\begin{aligned} u(x_k) =& 0, \\ \nabla u(x_k) =& 0, \\ \mathrm{Hess}_u(x_k) =& 0. \end{aligned}$$

Combining these, we have that the limit satisfies

$$\begin{aligned} \Delta u =& 0, \\ |\mathrm{Hess}_u|_\alpha \leq& 1, \\ u(x_\infty) =& 0, \\ \nabla u(x_\infty) =& 0, \\ \mathrm{Hess}_u(x_\infty) =& 0, \end{aligned} \tag{2.1}$$

and one of the following holds:

Case 1: $y_k \to y_\infty \neq x_\infty$. In this case $\mathrm{Hess}_u(y_\infty) > c_n|y_\infty - x_\infty|^\alpha \neq 0$.

Case 2: $y_k \to x_\infty$. In this case, we can rescale consider the rescaled sequence $\tilde{u}_k(x) = \frac{1}{|y_k - x_k|^{2+\alpha}} u_k(x_k + |y_k - x_k|x)$ defined in a ball of radius $\frac{R_k}{|y_k - x_k|}$ that also converges to an entire harmonic function satisfying (2.1) with 0 in place of x_∞ and that $\mathrm{Hess}_u(\frac{y_\infty - x_\infty}{|y_\infty - x_\infty|}) \neq 0$.

In either case above, we get an entire harmonic function that has distinct Hessian at 2 different points and that

$$\sup_{B_r} |u| \leq Cr^{2+\alpha} \leq C^{3-\varepsilon}, \varepsilon = \frac{1-\alpha}{2}.$$

On the other hand, by the Liouville Theorem (Corollary 1.3), u is a polynomial of degree at most 2, which is a contradiction to either case above that have non-constant Hessian! □

Next, we generalise the above interior Schauder estimates for Laplace operator to general elliptic operators.

Theorem 2.3 (Interior Schauder for elliptic operators)**.** *Let $U \subset \mathbb{R}^n$ be a bounded open domain and we consider an elliptic operator as in* (1.6) *$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ with $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \forall x \in U, \xi \in \mathbb{R}^n$ and $a^{ij}, c \in C^{0,\alpha}(U)$. If $u \in C^{2,\alpha}(U)$ satisfies*

$$Lu = f,$$

in an open domain $U \subset \mathbb{R}^n$, then for any pre-compact open subset $K \Subset U$ we have

$$\|u\|_{C^{2,\alpha}(K)} \leq C(\|Lu\|_{C^{0,\alpha}(U)} + \|u\|_{L^\infty(U)}) = C(\|f\|_{C^{0,\alpha}(U)} + \|u\|_{L^\infty(U)}),$$

for some constant $C = C(n, \alpha, K, U)$.

Proof. Again it suffices to prove the estimate for disks. We choose $K = B_r$ and $U = B_{2r}$ and then use covering argument to extend to general K, U as in the previous proposition.

Choose a fixed $x_0 \in B_r$. Since the coefficietns a^{ij}, c are Hölder continuous, we have

$$a^{ij}(x_0)\partial_i\partial_j u = Lu - (a^{ij}(x) - a^{ij}(x_0))\partial_i\partial_j u - \partial_i a^{ij}(x)\partial_j u - c(x)u.$$

By ellipticity of a^{ij} and compactness of B_{2r} we get

$$\Lambda|\xi|^2 \geq a^{ij}(x_0)\xi_i\xi_j \geq \lambda|\xi|^2.$$

And thus by the previous Proposition 2.1 (applied with $K = B_r, U = B_{2r}$) we have

$$\begin{aligned}[\mathrm{Hess}_u]_{C^{0,\alpha}(B_r)} \leq& C[\Delta u]_{C^{0,\alpha}(B_{2r})}\\ \leq& C(\lambda,\Lambda)[a^{ij}(x_0)\partial_i\partial_j u]_{C^{0,\alpha}(B_{2r})}\\ \leq& C(\lambda,\Lambda,\|a^{ij}\|_{C^{0,\alpha}(B_{2r})})\left([Lu]_{C^{0,\alpha}(B_{2r})} + r^\alpha[\mathrm{Hess}_u]_{C^{0,\alpha}(B_{2r})} + \|u\|_{C^2(B_{2r})}\right).\end{aligned}$$

By choosing $r < \frac{1}{2C(\lambda,\Lambda,\|a^{ij}\|_{C^{0,\alpha}(B_{2r})})}$ small enough we then have

$$[\mathrm{Hess}_u]_{C^{0,\alpha}(B_r)} \leq C\left([Lu]_{C^{0,\alpha}(B_{2r})} + \|u\|_{C^2(B_{2r})}\right). \tag{2.2}$$

Finally, we close the argument by applying the following interpolation inequality.

Lemma 2.4 (Interpolation inequality)**.**

For any $\varepsilon \in (0,1)$ there exists C_ε so that the following holds for any $u \in C^2(B_{2\rho})$ and $\rho \in (0,1)$:

$$\rho^2\|u\|_{C^2(B_\rho)} \leq \varepsilon\rho^{2+\alpha}\|u\|_{C^{2,\alpha}(B_{2\rho})} + C_\varepsilon\|u\|_{L^\infty(B_{2\rho})}. \tag{2.3}$$

Proof of Lemma. This is also proved by compactness (contradiction argument) using Arzela-Ascoli. See Problem set 2. □

We denote by

$$Q := \sup_{x\in B_2} \mathrm{dist}(x,\partial B_2)^2|\mathrm{Hess}_u(x)|,$$

and notice that the supremum is attained in the interior $(\mathrm{dist}(x,\partial B_2)^2|D^2u(x)| = 0 \quad \text{on } \partial B_2)$ for some $x_0 \in B_2$. Let $\rho = \frac{1}{3}\mathrm{dist}(x_0,\partial B_2)$, we get $B_{2\rho}(x_0) \subset B_2$ and

$$\begin{aligned}Q =& 9\rho^2|\mathrm{Hess}_u(x_0)|\\ \leq& 9\rho^2\|\mathrm{Hess}_u\|_{L^\infty(B_\rho(x_0))}\\ \leq& 9\varepsilon\rho^{2+\alpha}\|u\|_{C^{2,\alpha}(B_{2\rho}(x_0))} + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))} \quad \text{by (2.3)}\\ \leq& 9\varepsilon\rho^{2+\alpha}\left([\mathrm{Hess}_u]_{C^{0,\alpha}(B_{2\rho}(x_0))} + \|u\|_{C^2(B_{2\rho}(x_0))}\right) + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))}\\ \leq& 9\varepsilon\rho^{2+\alpha}\left[C\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + C\|u\|_{C^2(B_{2\rho}(x_0))}\right] + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))} \quad \text{by (2.2)}\\ \leq& 9\varepsilon\rho^{2+\alpha}\left[C\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + C[\mathrm{Hess}_u]_{C^{0,\alpha}(B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C_\varepsilon\|u\|_{L^\infty(B_{2\rho}(x_0))}\end{aligned}$$

$$\begin{aligned}
&\leq (9\varepsilon C + 9C_\varepsilon)\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C\varepsilon\rho^2[\mathrm{Hess}_u]_{L^\infty(B_{2\rho}(x_0))} \\
&\leq \tilde{C}(\varepsilon, C)\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C\varepsilon \sup_{x\in B_{2\rho}(x_0))} \mathrm{dist}(x,\partial B_2)^2[\mathrm{Hess}_u]_{L^\infty(B_{2\rho}(x_0))} \\
&\leq \tilde{C}\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] + 9C\varepsilon Q.
\end{aligned}$$

By choosing $\varepsilon < \frac{1}{18C}$ and absorbing the second term on the right hand side to the left, we get

$$Q \leq \tilde{C}\left[\|Lu\|_{C^{0,\alpha}((B_{2\rho}(x_0))} + \|u\|_{L^\infty(B_{2\rho}(x_0))}\right] \leq \tilde{C}[\|Lu\|_{C^{0,\alpha}(B_2)} + \|u\|_{L^\infty(B_2)}].$$

Applying (2.2) on compact subsets of bounded open sets (could easily see by covering argument), we get

$$\begin{aligned}
|u\|_{C^{2,\alpha}(B_1)} =& |u\|_{C^2(B_1)} + [\mathrm{Hess}_u]_{C^{0,\alpha}(B_1)} \\
\leq& |u\|_{C^2(B_1)} + C[\|Lu\|_{C^{0,\alpha}(B_{\frac{3}{2}})} + \|u\|_{C^2(B_{\frac{3}{2}})}] \\
\leq& C_1[\|Lu\|_{C^{0,\alpha}(B_{\frac{3}{2}})} + \|u\|_{L^\infty(B_{\frac{3}{2}})} + 2(\frac{3}{2})^2[\mathrm{Hess}_u]_{L^\infty(B_{\frac{3}{2}r}(x_0))}] \\
\leq& C_2[\|Lu\|_{C^{0,\alpha}(B_{\frac{3}{2}})} + \|u\|_{L^\infty(B_{\frac{3}{2}})} + Q] \\
\leq& \tilde{C}[\|Lu\|_{C^{0,\alpha}(B_2)} + \|u\|_{L^\infty(B_2)}].
\end{aligned}$$

□

3. Existence theory for elliptic PDEs

We treat the existence theory and regularity theory separately in many PDE problems. First we need to obtain the existence of weak solution in some larger space, and then we proceed to prove the regularity / partial regularity of the solution. There are several approaches using abstract tools from functional analysis or topological arguments. We will present the Lax-Milgram theorem for existence of solutions to linear PDE and a fixed point argument that can be applied to non-linear problems.

3.1. **Lax-Milgram and weak solutions.** Consider an elliptic operator of divergence form defined before,

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u,$$

i.e. the coefficients satysfying

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \lambda > 0.$$

W define the bilinear form associated to L by

$$B_L[u,v] =: \int_U u \cdot Lv dx = \int_U -a^{ij}(x)u_{x_i}v_{x_j} + c(x)u(x)v(x)dx \tag{3.1}$$

for $u, v \in H_0^1(U), U \subset \mathbb{R}^n$.

Definition 3.1. A weak solution $u \in H^1(U)$ of the Dirichlet problem

$$\begin{aligned} Lu =& f \\ u|_{\partial U} =& 0, \end{aligned} \tag{3.2}$$

if

$$B_L[u,v] = \langle f, v\rangle_{L^2(U)}, \quad \forall v \in H_0^1(U). \tag{3.3}$$

The following functional analytic theorem will give existence of weak solutions.

Theorem 3.2 (Lax-Milgram)**.** *Let H be a real Hilbert space and B a bilinear form on H:*

$$B : H \times H \to \mathbb{R},$$

satisfying

$$|B[u,v]| \leq \alpha\|u\|\|v\|, \quad \forall u, v \in H, \tag{3.4}$$

$$\beta\|u\|^2 \leq B[u,u], \quad \forall u \in H, \tag{3.5}$$

for some $\alpha, \beta > 0$. Then for any $f \in H$, there exists a unique $u \in H$ so that

$$B[u,v] = \langle f, v\rangle.$$

Sketch of Proof: The existence of u is a consequence of Riesz Representation theorem for bounded (by (3.4)) linear functional, and the uniqueness part follows from the coercivity (3.4). □

Indeed, we can characterise exactly when the coercivity condition in the Lax-Milgram theorem fails.

Theorem 3.3 (Fredholm alternative)**.** *For an elliptic operator L satisfying the conditions stated at the beginning of the section, one of the following items hold:*

- *For each $f \in L^(U)$, there eixsts a unique weak solution u of the boundary value problem*
$$\begin{cases} Lu = f, & in\ U \\ u = 0, & on\ \partial U \end{cases}.$$
- *There exists a nontrivial weak solution $u \not\equiv 0$ to the homogenous problem*
$$\begin{cases} Lu = 0, & in\ U \\ u = 0, & on\ \partial U \end{cases}.$$

Indeed, when $c(x) \leq 0$, the first case always happens.

Corollary 3.4. *If the elliptic operator $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ satisfies $c(x) \leq 0$, then the first case in Fredholm alternative holds.*

Proof. The second case does not happen, because the weak maximum principle proved in Lecture 1 (for $c \leq 0$) guarantees the uniqueness of solutions. □

We can also deal with the case with non-zero boundary data.

Proposition 3.5. *Suppose L is an elliptic operator $Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$ satisfying $c(x) \leq 0$ and $U \subset \mathbb{R}^n$ is a smooth bounded domain. Then the following boundary value problem has a unique solution for any $f \in C^{0,\alpha}(U), \phi \in C^{2,\alpha}(U)$:*

$$\begin{cases} Lu = f, & in\ U \\ u = \phi, & on\ \partial U \end{cases}. \tag{3.6}$$

Proof. We can extend ϕ to a $\Phi \in C^{2,\alpha}(U)$ so that $\Phi|_{\partial U} = \phi$, as U is smooth.Then by the previous Corollary, there exists a unique solution for the following problem
$$\begin{cases} Lv = f - L\Psi, & \text{in } U \\ v = 0, & \text{on } \partial U \end{cases}.$$
Then we see that $f - L\Psi \in C^{0,\alpha}(U) \subset L^2(U)$ because the operator is second order, and thus $u = v + \Psi$ is a solution (3.6). □

Combining the existence of a weak solution (3.3) in H^1_0 for linear elliptic equations, we can apply the De Giorgi Nash Moser Moser theory in Lecture 4 to gain L^∞ and Hölder regularity, and then apply Schauder theory to get higher regularity for such solutions.

3.2. **Leray-Schauder existence theory.** The second existence theory is based on fixed point argument and can be applied to quasi-linear PDEs, e.g. minimal surface equation.

Recall the classical Brouwer's fixed point theorem, which states that "a continuous map from closed unit ball in $\mathbb{R}^n$ to itself must have have a fixed point", can be generalised to maps of compact convex sets of Banach spaces.

Theorem 3.6 (Schauder's fixed point theorem, Generlised Brouwer's)**.** *Let K be a compact convex set in a Banach space $\mathcal{B}$ and let $T : K \to K$ be continuous. Then T has a fixed point.*

As a corollary, one gets

Corollary 3.7. *Let $\mathcal{B}$ be a Banach space and $B \subset \mathcal{B}$ is its open unit ball. Suppose $T : \bar{B} \to \mathcal{B}$ is a continuous map such that*

- *The map T is compact, i.e. images of any compact set is precompact.*
- *$T(\partial B) \subset B$.*

Then T has a fixed point.

Sketch of Proof: Apply the Schauder's fixed point theorem to the map $T^\star : \bar{B} \to \bar{B}$ defined by

$$T^\star(x) = \begin{cases} T(x), & \text{for } \|T(x)\| \leq 1 \\ \frac{T(x)}{\|T(x)\|}, & \text{for } \|T(x)\| \geq 1 \end{cases},$$

and notice that the fixed point cannot happen at ∂B because $|T(y)| < 1 = |y|, \forall y \in \partial B$. □

The fixed point theorem can be

Theorem 3.8 (Leray-Schauder fixed point theorem)**.** *Let $\mathcal{B}$ be a Banach space and*

$$T : \mathcal{B} \times [0, 1] \to \mathcal{B}$$

a compact map such that:

- *$T(x, 0) = 0$ for each $x \in \mathcal{B}$;*
- *There exists a constant $M > 0$ so that for each $(x, t) \in \mathcal{B} \times [0, 1]$ which satisfies $x = T(x, t)$, there holds $\|x\| < M$.*

Then there is a fixed point $y \in \mathcal{B}$ of the map $T(\cdot, 1) : \mathcal{B} \to \mathcal{B}$ given by $T(y, 1) = y$.

Proof. Without loss of generality, we may assume $M = 1$. Otherwise one can just rescale the norm on Banach space by a factor of $\frac{1}{M}$ and notice that a fixed point is unchanged by this norm scaling. For any $\varepsilon \in (0, 1)$, we define a map from the closed unit ball,

$$T_\varepsilon : \bar{B} \to \mathcal{B}$$
$$T_\varepsilon(x) =: \begin{cases} T(\frac{x}{\|x\|}, \frac{1-\|x\|}{\varepsilon}), & \text{if } 1 - \varepsilon \leq \|x\| \leq 1 \\ T(\frac{x}{1-\varepsilon}, 1), & \text{if } \|x\| \leq 1 - \varepsilon \end{cases}.$$

For each ε, we see the image of ∂B by T_ε is

$$T_\varepsilon(\partial B) = T(\partial B, \frac{1-1}{\varepsilon}) = T(\partial B, 0) = 0,$$

by the definition of $T(\cdot, 0)$. So the Corollary 3.7 implies that there is a fixed point x_ε of T_ε for any ε. We define further that

$$t_\varepsilon =: \begin{cases} \frac{1-\|x_\varepsilon\|}{\varepsilon}, & \text{if } 1 - \varepsilon \leq \|x_\varepsilon\| \leq 1 \\ 1, & \text{if } \|x_\varepsilon\| \leq 1 - \varepsilon \end{cases},$$

which is the second parameter of T_ε for this fixed point.

By compactness of T, we can find a subsequence so that

$$(x_{\varepsilon_k}, t_{\varepsilon_k}) \to (\hat{x}, \hat{t}) \in \bar{B} \times [0,1].$$

There are 2 possible cases:

- If $t < 1$, then for ε_k small enough, there holds $t_{\varepsilon_k} < 1$, and thus
$$\|x_{\varepsilon_k}\| \geq 1 - \varepsilon_k \to 1 = \|x\|.$$
But this is a contradiction to the second condition that $\|x\| < M = 1$ as a fixed point $x = T(x,t)$.
- If $t = 1$, and thus $x = T(x,1)$ gives a fixed point for $T(\cdot, 1)$ as desired.

□

We want to apply the Leray-Schauder theorem to the existence theory of quasi-linear elliptic PDEs of the form:

$$\partial_i(a^{ij}(x,u,\nabla u)\partial_j u) + c(x)u = 0, \quad \text{in } U \subset \mathbb{R}, \tag{3.7}$$

where the coefficients a^{ij}, c are $C^{0,\alpha}$ about every components of their variable and $c(x) \leq 0$.

Theorem 3.9 (Quasi-linear existence)**.** *Let $\alpha \in (0,1)$, $U \subset \mathbb{R}^n$ a bounded smooth open domain and $\phi \in C^{2,\alpha}(\bar{U})$. Suppose further that for some $\beta \in (0,1)$, there exists $M > 0$ constant so that the following holds: For every $t \in [0,1]$, each $C^{2,\alpha}$ solution u (not assuming it exists) of*

$$\begin{cases} \partial_i(a^{ij}(x,u,\nabla u)\partial_j u) + c(x)u = 0 & \text{in } U, c \leq 0, \\ u = t\phi & \text{on } \partial U, \end{cases} \tag{3.8}$$

satisfies the a priori estimate

$$\|u\|_{C^{1,\beta}(\bar{U})} < M.$$

Then the Dirichlet problem

$$\begin{cases} \partial_i(a^{ij}(x,u,\nabla u)\partial_j u) + c(x)u = 0 & \text{in } U, \\ u = \phi & \text{on } \partial U \end{cases} \tag{3.9}$$

has a solution in $C^{2,\alpha}(\bar{U})$.

Proof. We define an operator

$$\begin{aligned} T :& C^{1,\beta}(\bar{U}) \times [0,1] \to C^{1,\beta}(\bar{U}) \\ T(v,t) =& u, \end{aligned}$$

where $u = T(v,t)$ is the unique solution of the linear problem obtained by replacing $u, \nabla u$ with $v, \nabla v$ in the coefficients a^{ij}.

$$\begin{cases} \partial_i(a^{ij}(x,v,\nabla v)\partial_j u) + tc(x)u = 0 & \text{in } U, \\ u = t\phi & \text{on } \partial U, \end{cases}$$

We see that any solution u of (3.8) is a fixed point of T. And by the assumption of the theorem, any such fixed point $u = T(u,t), t \in [0,1]$ must satisfies

$$\|v\|_{C^{1,\beta}(\bar{U})} < M.$$

So the Leray-Schauder fixed point theorem implies the existence of a fixed point for the map $T(\cdot, 1)$, namely a solution to (3.9).

The $C^{2,\alpha}$ regularity of the solution is coming from Schauder estimates, as the coefficients are now in $C^{0,\beta}$ (as u is in $C^{1,\beta}$). □

3.3. **Example: Existence of solution to minimal surface equation.** The area of graph of u over $U \subset \mathbb{R}^n$ is

$$A_u(U) = \int_U \sqrt{1 + |\nabla u|^2} dx.$$

For any compactly supported variation $\phi \in C_0^\infty(U)$ we have the first variation formula

$$\begin{aligned}
0 =& \frac{d}{dt}|_{t=0} A_{u+t\phi}(U) \\
=& \int_U \frac{d}{dt} \sqrt{1 + |\nabla u + t\nabla\phi|^2} dx|_{t=0} \\
=& \int_U \frac{1}{2\sqrt{1 + |\nabla u + t\nabla\phi|^2}} 2\langle \nabla u + t\nabla\phi, \nabla\phi\rangle dx|_{t=0} \\
=& \int_U \langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nabla\phi\rangle dx \\
=& -\int_U \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \phi dx.
\end{aligned}$$

So the Euler-Lagrange equation of the area functional is the minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{3.10}$$

To apply the Leray-Schauder estimate above, we need to prove a priori gradient estimates.

Lemma 3.10. *Let $U \subset \mathbb{R}^n$ be a mean convex bounded smooth domain and $\phi \in C^\infty(\bar{U})$. Then any solution to the boundary value minimal surface equation:*

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) =& 0 \quad \text{in } U \\
u =& \phi \quad \text{on } \partial U,
\end{aligned} \tag{3.11}$$

satisfies the gradient estimate

$$\|\nabla u\|_{C^{0,\beta}(\bar{U})} < M,$$

for some $M > 0$ and $\beta \in (0,1)$.

The proof of this Hölder bounds of gradient will use De Goirgi - Nash - Moser theory, which is the material of next lecture.

As a application of Theorem 3.9 and Lemma 3.10, we have

Theorem 3.11. *Let $U \subset \mathbb{R}^n$ and ϕ satisfy the same conditions as in Lemma 3.10. There exists a unique smooth solution to* (3.11)*.*

Remark 3.12. One gets $C^{2,\alpha}$ regularity by Leray-Schauder, then the higher regularity follows from Schauder estimates from Lecture 2.

Remark 3.13. This is actually a minimising solution (i.e. the surface with least area with that prescribed boundary). In general, minimising hypersurfaces with prescribe boundary are only smooth up to dimension 8 (and in higher dimension may have singular set of codimension 8). However, when they are graphical, we know that they are actually smooth in all dimensions.

3.4. **Variational method.** Another approach to the existence of minimal surface is through a variational method. Let u_k be a minimizing sequence of the area functional so that

$$A_{u_k}(U) \to \inf_{u \in C^\infty(U), u|_{\partial U} = \phi} A_u(U).$$

In order to get a subsequence converging to a limit, we need some estimates and compactness in function space.

Since the minimising sequence have uniformly bounded area, we have

$$\int_U |\nabla u_k| \leq \int_U \sqrt{1 + |\nabla u_k|^2} \leq A_0.$$

And by Poincare inequality and that $u_k - \Phi = 0$ on ∂U (where Ψ is an extension of ϕ to the interior of U), we get

$$\int_U |u_k - \Psi| \leq \int_U |\nabla u_k| \leq A_0,$$

and so

$$\int_U |u_k| \leq C(A_0, \phi).$$

Thus the sequence is uniformly bounded in $W^{1,1}(U)$. Sobolev inequality gives that $W^{1,1} \mapsto L^{\frac{n}{n-1}}$ compactly. So the existence of a solution in L^p for $p \leq \frac{n}{n-1}$ follows by taking a subsequential limit

$$u_{k_k} \to u_0.$$

For higher regularity, we need to Apply De Goirgi Nash - Moser theory from next lecture and Schauder theory from the last lecture.

4. De Giorgi Nash-Moser theory

The De Giorgi-Nash-Moser theory in elliptic PDE is providing the initial L^∞ and $C^{0,\alpha}$ regularity, before higher regularity theory like Schauder theory applies. We will present here the Moser's approach for getting the local boundedness (which is iteration on the exponents of L^p norms, while De Giorgi's approach is iteration on the super-level sets).

4.1. Initial L^∞ bound and Moser iteration.

Let

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + c(x)u$$

be an elliptic operator of divergence form and $U \subset \mathbb{R}^n$ a bounded open domain as before. Suppose the coefficients satisfies

$$\|a^{ij}\|_{L^\infty(U)}, \|c\|_{L^q(U)} \leq L_0$$

and

$$\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi^2|, \lambda, \Lambda > 0. \tag{4.1}$$

Our first theorem is the local boundedness of u, which only requires L^p boundedness of the coefficients and inhomogeneous term.

We will present a simplified case with $c = 0$ for the homogeneous equation, the general case follows by exactly the same argument with more technicality involved in absorbing the extra terms.

Theorem 4.1 ($L^p \to L^\infty$ estimate)**.** *Suppose $u \in H^1(B_1)$ is a subsolution, i.e.*

$$\int_{B_1} a^{ij} D_i u D_j \phi \leq 0, \quad \forall \phi \in H_0^1(B_1), \phi \geq 0. \tag{4.2}$$

Then for any $\theta \in (0,1)$, we have in the smaller ball B_θ that

$$\sup_{B_\theta} u^+ \leq C\frac{\|u^+\|_{L^p(B_1)}}{(1-\theta)^{\frac{n}{p}}},$$

for some positive constant $C = C(n.\lambda, \Lambda, p, q)$.

Proof. For $k > 0$ to be determined and $m \in \mathbb{N}$, we define

$$\bar{u} = u^+ + k,$$

and

$$\bar{u}_m = \begin{cases} \bar{u}, & u < m \\ k+m, & u \geq m \end{cases}.$$

Then one observes

$$k \leq \bar{u}_m \leq m + k,$$

and

$$\bar{u}_m \equiv \text{Constant}, D\bar{u}_m = 0, \quad \text{when } u < 0 \text{ or } u > m.$$

We choose the following non-netative test function

$$0 \le \phi = \zeta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1),$$

for some $\beta \ge 0, \zeta \in C_0^1(B_1)$.

We compute

$$\begin{aligned} D\phi =& 2\zeta D\zeta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + \zeta^2 \beta \bar{u}_m^{\beta-1} D\bar{u}_m \bar{u} + \zeta^2 \bar{u}_m^\beta D\bar{u} \\ =& 2\zeta D\zeta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + \zeta^2 \bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}), \end{aligned}$$

where we used that $\bar{u}_m = \bar{u}$ when $D\bar{u}_m \ne 0$. Plugging into the equation (4.2) we get

$$\begin{aligned} (4.3)\quad 0 \ge& \int_{B_1} a^{ij} D_i u D_j \phi \\ =& \int_{B_1 \cap \{u>0\}} a^{ij} D_i u D_j \phi \\ =& \int_{B_1 \cap \{u>0\}} a^{ij} D_i u \cdot 2\zeta D\zeta(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) + a^{ij} D_i u \cdot \zeta^2 \bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) \\ \ge& \int_{B_1 \cap \{u>0\}} -\Lambda|D\bar{u}| \cdot 2\zeta|D\zeta| \cdot \bar{u}_m^\beta \bar{u} + \lambda\beta\zeta^2|D\bar{u}_m|^2 \bar{u}_m^\beta + \lambda\zeta^2|D\bar{u}|^2 \bar{u}_m^\beta \\ \ge& \int_{B_1 \cap \{u>0\}} [-\frac{1}{2}\lambda\zeta^2|D\bar{u}|^2\bar{u}_m^\beta - \frac{1}{2}\frac{\Lambda^2}{\lambda} \cdot 4|D\zeta|^2 \bar{u}_m^\beta \bar{u}^2] + \lambda\beta\zeta^2|D\bar{u}_m|^2\bar{u}_m^\beta + \lambda\zeta^2|D\bar{u}|^2\bar{u}_m^\beta \\ & \text{(Here we used Cauchy-Schwarz for the first term in the previous line)} \\ =& \int_{B_1 \cap \{u>0\}} -2\frac{\Lambda^2}{\lambda} \cdot |D\zeta|^2 \bar{u}_m^\beta \bar{u}^2 + \lambda\beta\zeta^2|D\bar{u}_m|^2\bar{u}_m^\beta + \frac{1}{2}\lambda\zeta^2|D\bar{u}|^2\bar{u}_m^\beta. \end{aligned}$$

Thus

$$\beta \int_{B_1} \zeta^2|D\bar{u}_m|^2\bar{u}_m^\beta + \int_{B_1} \zeta^2|D\bar{u}|^2\bar{u}_m^\beta \le C\int_{B_1} |D\zeta|^2\bar{u}_m^\beta\bar{u}^2, \tag{4.4}$$

for some $C = C(\beta, \Lambda, \lambda)$.

(Notice that if $c \ne 0$ or there is an inhomogeneous term for the equation, then the LHS of (4.3) is not zero and one needs a few more steps in the absorption of terms.)

We see the RHS (4.4) is "roughly" an L^2 norm of the function

$$w = \bar{u}_m^{\frac{\beta}{2}} \bar{u},$$

whose L^2 norm of derivative is "roughly" bounded by the LHS of (4.4) as follows

$$\begin{aligned} |Dw|^2 =& |\frac{\beta}{2}\bar{u}_m^{\frac{\beta}{2}-1}\bar{u}\nabla\bar{u}_m + \bar{u}_m^{\frac{\beta}{2}}\nabla\bar{u}|^2 \\ =& |\frac{\beta}{2}\bar{u}_m^{\frac{\beta}{2}}\nabla\bar{u}_m + \bar{u}_m^{\frac{\beta}{2}}\nabla\bar{u}|^2 \\ \le& (1+\beta)(\beta|D\bar{u}_m|^2\bar{u}_m^\beta + |D\bar{u}|^2\bar{u}_m^\beta). \end{aligned}$$

Namely (4.4) reads

$$\int_{B_1} |Dw|^2\zeta^2 \leq (1+\beta)\int_{B_1} |D\zeta|^2 w^2.$$

By Sobolev inequality applied to the compactly supported ζw, we get

$$[\int_{B_1} |\zeta w|^{\frac{2n}{n-2}}]^{\frac{n-2}{2n}} \leq [\int_{B_1} |D(\zeta w)|^2]^{\frac{1}{2}} \leq \int_{B_1} |Dw|^2\zeta^2 + \int_{B_1} |D\zeta|^2 w^2]^{\frac{1}{2}} \leq [(2+\beta)\int_{B_1} |D\zeta|^2 w^2]^{\frac{1}{2}}.$$

Now we choose the cut-off function $\zeta \in C_0^1(B_1)$ so that for $0 < r < R \leq 1$ there holds

$$\zeta \equiv 1, \quad \text{in } B_r$$
$$0 \leq \zeta \leq 1, \quad \text{in } B_R$$
$$|D\zeta| \leq \frac{2}{R-r}.$$

Noticing $\bar{u}_m \leq \bar{u}$ and $\zeta \equiv 1$ in B_r, we obtain

$$\begin{aligned}
[\int_{B_r} \bar{u}_m^{(\beta+2)\frac{n}{n-2}}]^{\frac{n-2}{2n}} =&[\int_{B_r} \bar{u}_m^{\frac{\beta+2}{2}\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\\
\leq&[\int_{B_r} (\bar{u}_m^{\frac{\beta}{2}}\bar{u})^{\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\\
\leq&[\int_{B_r} w^{\frac{2n}{n-2}}]^{\frac{n-2}{2n}}\\
\leq&(\int_{B_1} |\zeta w|^{\frac{2n}{n-2}})^{\frac{n-2}{2n}}\\
\leq&[(2+\beta)\int_{B_1} |D\zeta|^2 w^2]^{\frac{1}{2}}\\
\leq&\frac{2\sqrt{2+\beta}}{(R-r)}[\int_{B_1} w^2]^{\frac{1}{2}}\\
\leq&\frac{2\sqrt{2+\beta}}{(R-r)}[\int_{B_1} \bar{u}^{\beta+2}]^{\frac{1}{2}}.
\end{aligned}$$

By letting $m \to \infty$, we can replace $\bar{u}_m$ by $\bar{u}$ and so

$$\|\bar{u}\|_{L^{p\chi}(B_1)} \leq [\frac{2(p-1)}{(R-r)^2}]^{\frac{1}{p}}\|\bar{u}\|_{L^p B_R},$$

where $p = \beta+2, \chi = \frac{n}{n-2}$.

Here $p\chi > \gamma$ so we get an improvement from L^p to $L^{p\chi}$, as $p\chi > p$. After iterating i times with

$$p_i = 2\chi^i \to \infty, r_i = \theta + \frac{1}{2^i}(1-\theta) \to \theta, \quad i \in \mathbb{N},$$

we get

$$\|\bar{u}\|_{L^{p_i}(B_{r_i})} \leq [\frac{4^i \cdot 2(p_i-1)}{(1-\theta)^2}]^{\frac{1}{p_i}}\|\bar{u}\|_{L^{p_{i-1}}(r_{i-1})}$$

$$\leq \Pi_{k=1}^{i}\left[4^{i}\cdot 2(p_i-1)\right]^{\frac{1}{p_i}}\cdot \Pi_{k=1}^{i}\left[\frac{1}{(1-\theta)^2}\right]^{\frac{1}{p_i}}\cdot \|\bar{u}\|_{L^2(B_1)}.$$

Taking the limit we get

$$\|\bar{u}\|_{L^\infty(B_\theta)} \leq C\cdot \frac{1}{(1-\theta)^2}\|\bar{u}\|_{L^2(B_1)}$$

□

4.2. **Harnack inequality and Hölder regularity.** As a consequence of Theorem 4.1, we also have Inf bound for non-negative super solutions.

Theorem 4.2. *Suppose $u \in H^1(B_1)$ is a non-negative supersolution, i.e.*

$$\int_{B_1} a^{ij}D_iuD_j\phi \geq 0, \quad \forall \phi \in H_0^1(B_1), \phi \geq 0. \tag{4.5}$$

Then for any $\theta \in (0,1), p \leq \frac{n}{n-2}$, we have in the smaller ball B_θ that

$$\inf_{B_\theta} u \geq C\|u\|_{L^p(B_1)},$$

for some positive constant $C = C(n.\lambda, \Lambda, p, q)$.

Idea of Proof: Apply Theorem 4.1 to $u^{-\beta}$ (so that a super-solution becomes a subsolution). The detail is left as an exercise. □

So combining Theorem 4.1 and Theorem 4.2, we have the Harnack inequality for actual non-negative solutions.

Theorem 4.3 (Moser-Harnack Inequality)**.** *Suppose $u \in H^1(B_R)$ is a non-negative weak solution, i.e.*

$$\int_{B_R} a^{ij}D_iuD_j\phi = 0, \quad \forall \phi \in H_0^1(B_R), \phi \geq 0. \tag{4.6}$$

We have

$$\sup_{B_{\frac{R}{2}}} u \leq C \inf_{B_{\frac{R}{2}}} u,$$

for some uniform constant $C = C(n, \lambda, \Lambda)$.

The Moser-Harnack inequality will then give us oscillation decay when the radius of balls shrinks, providing Hölder regularity (by standard iteration argument of Campanato).

Theorem 4.4 (Hölder continuity of weak solutions)**.** *If $u \in H^1(U)$ is a weak solution to $Lu = 0$ in a bounded open domain $U \subset \mathbb{R}^n$, then for any $B_R(x_0) \Subset \Omega$ we have the following:*

- *For all $r \leq R$ we have*

$$\mathrm{osc}_{B_r(x_0)} u \leq C\left(\frac{r}{R}\right)^\alpha \mathrm{osc}_{B_R(x_0)} u,$$

for some $\alpha = \mu(n, \lambda, \Lambda) \in (0,1)$ and $C = C(n, \lambda, \Lambda) > 0$. Here

$$\mathrm{osc}_{B_\rho} u = \sup_{B_\rho} u - \inf_{B_\rho} u.$$

- *u is Hölder in a smaller ball, with the estimate*

$$R^\alpha \,[u]_{C^{0,\alpha}\left(B_{R/4}(x_0)\right)} \le C\|u\|_{L^\infty(B_R)}.$$

Proof. First we prove the oscillation decay. Let

$$M := \sup_{B_R} u, \qquad m := \inf_{B_R} u, \qquad \omega := M - m.$$

Define

$$v := u - m \ge 0, \qquad w := M - u \ge 0 \quad \text{in } B_R.$$

Since the operator is linear and $Lu = 0$, we have $Lv = Lw = 0$.

By Moser's Harnack inequality, there exists $C_H > 1$ such that

$$\sup_{B_{R/2}} v \le C_H \inf_{B_{R/2}} v, \qquad \sup_{B_{R/2}} w \le C_H \inf_{B_{R/2}} w.$$

Set

$$A := \sup_{B_{R/2}} u, \qquad B := \inf_{B_{R/2}} u.$$

Applying Harnack to $v = u - m$ gives

$$A - m \le C_H(B - m),$$

and applying it to $w = M - u$ gives

$$M - B \le C_H(M - A).$$

Adding the above 2 bounds we get

$$(A - m) + (M - B) \le C_H\big[(B - m) + (M - A)\big].$$

Using

$$(A - m) + (M - B) = \omega + (A - B), \qquad (B - m) + (M - A) = \omega - (A - B),$$

we obtain

$$\omega + (A - B) \le C_H\big(\omega - (A - B)\big).$$

Rearranging,

$$(C_H + 1)(A - B) \le (C_H - 1)\omega,$$

hence

$$\operatorname{osc}_{B_{R/2}} u = A - B \le \frac{C_H - 1}{C_H + 1}\,\omega.$$

Setting $\sigma = \dfrac{C_H - 1}{C_H + 1} \in (0,1)$ completes the proof of first part on oscillation decay.

From the oscillation decay theorem, there exists $\sigma \in (0,1)$ such that

$$\operatorname{osc}_{B_{R/2}} u \le \sigma \operatorname{osc}_{B_R} u.$$

Iterating this estimate, we obtain for all $k \in \mathbb{N}$,

(1)
$$\operatorname{osc}_{B_{R/2^k}} u \le \sigma^k \operatorname{osc}_{B_R} u.$$

Choose $\alpha > 0$ such that

$$\sigma = 2^{-\alpha}, \quad \text{i.e.} \quad \alpha = -\frac{\log \sigma}{\log 2}.$$

Let $0 < \rho \le R$ and choose $k \in \mathbb{N}$ satisfying

$$\frac{R}{2^{k+1}} < \rho \le \frac{R}{2^k}.$$

Then, by monotonicity of oscillation in the radius,

$$\operatorname{osc}_{B_\rho} u \le \operatorname{osc}_{B_{R/2^k}} u \le \sigma^k \operatorname{osc}_{B_R} u = 2^{-\alpha k} \operatorname{osc}_{B_R} u.$$

Since $\rho \le R/2^k$, we have $2^{-k} \le \rho/R$, and hence

$$\operatorname{osc}_{B_\rho} u \le \left(\frac{\rho}{R}\right)^\alpha \operatorname{osc}_{B_R} u.$$

Finally, for any $x, y \in B_{R/2}$, setting $\rho = |x - y|$ yields

$$|u(x) - u(y)| \le \operatorname{osc}_{B_\rho} u \le C|x - y|^\alpha,$$

which proves $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$.

□

5. Regularity theory of harmonic maps

Let $u : \Omega \subset \mathbb{R}^n \to (N, h)$ be a map into a compact Riemannian manifold N (isometrically embedded in $\mathbb{R}^k$). The *Dirichlet energy* is

$$E(u; \Omega) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx,$$

and its critical points under compactly supported variations are the *harmonic maps*. In extrinsic form (with $N \hookrightarrow \mathbb{R}^k$), the Euler–Lagrange equation becomes

$$\Delta u = A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of $N \subset \mathbb{R}^k$.

Let $u : \Omega \subset \mathbb{R}^n \to N$ be a smooth harmonic map, and set

$$e := |\nabla u|^2.$$

Then the Bochner identity for harmonic maps gives

$$\Delta e = 2|\nabla^2 u|^2 - 2\langle R^N(\nabla u, \nabla u)\nabla u, \nabla u\rangle, \tag{5.1}$$

where R^N denotes the Riemann curvature tensor of the target manifold N.

Since N is compact, its curvature is bounded, and hence

$$-\Delta e \le C\, e^2 \qquad \text{in } \Omega, \tag{5.2}$$

for a constant $C > 0$ depending only on the geometry of N. The inequality (5.2) is understood in the weak (distributional) sense for $u \in W^{1,2}$.

5.1. **Epsilon-regularity.**

Theorem 5.1 (Epsilon-regularity in dimension 2)**.** *There exists* $\varepsilon_0 > 0$ *such that the following holds. If* $u : B_1 \subset \mathbb{R}^2 \to N$ *is a weak harmonic map satisfying*

$$\int_{B_1} |\nabla u|^2 \leq \varepsilon_0,$$

then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C \left(\int_{B_1} |\nabla u|^2 \right)^{1/2},$$

where C *depends only on the geometry of* N*. In particular,* $u \in C^\infty(B_{1/2})$*.*

Proof. We want to apply Moser iteration to (5.1)

Fix $0 < \rho < R \leq 1$, and let $\eta \in C_c^\infty(B_R)$ satisfy

$$\eta \equiv 1 \text{ on } B_\rho, \qquad 0 \leq \eta \leq 1, \qquad |\nabla \eta| \leq \frac{2}{R-\rho}.$$

For $p \geq 1$, multiply (5.1) by $\eta^2 e^{p-1}$ and integrate by parts, we get

$$\begin{aligned} \int \Delta e \eta^2 e^{p-1} &\leq \int C\eta^2 e^{p+1} \\ \int 2\eta e^{p-1} \langle \nabla e, \nabla \eta \rangle + \int (p-1)\eta^2 e^{p-2} |\nabla e|^2 &\leq \int C\eta^2 e^{p+1} \\ -2\int |\nabla e||\nabla \eta| \eta e^{p-1} + \int (p-1)\eta^2 e^{p-2}|\nabla e|^2 &\leq \int C\eta^2 e^{p+1} \\ -\int \frac{2}{p-1} |\nabla \eta|^2 e^p - \int \frac{p-1}{2} \eta^2 e^{p-2} |\nabla e|^2 + \int (p-1)\eta^2 e^{p-2}|\nabla e|^2 &\leq \int C\eta^2 e^{p+1} \end{aligned}$$

Namely

$$\int |\nabla(\eta e^{p/2})|^2 \leq C \int |\nabla \eta|^2 e^p + C \int \eta^2 e^{p+1}. \tag{5.3}$$

By Sobolev embedding $W^{1,2} \mapsto L^q$ compactly for $q = \frac{2n}{n-2}$, we have

$$\|e\|_{L^{\frac{p}{2}\cdot\frac{2n-\delta}{n-2}}(B_r)} \leq C_\eta \|e\|_{L^{p+1}(B_R)}.$$

In dimension $n = 2$, this directly implies L^q bound of u for any $q > 1$

In dimension $n \geq 3$, by choosing $\delta < \frac{4p+4-2n}{p}$ we get an improvement as

$$\frac{p}{2} \cdot \frac{2n-\delta}{n-2} > p+1.$$

Moser iteration then implies L^∞ bound of e by L^{p+1} bound of e. (Although the energy bound only implies L^1 norm of e, so we don't a priori have L^{p+1} bound of u to get L^∞ by the Dirichlet energy.)

□

This can be generalized to the case of minimizing harmonic maps in higher dimensions.

Theorem 5.2 (Schoen–Uhlenbeck ε–regularity)**.** *Let $n \geq 3$ and let N be a compact Riemannian manifold. There exist constants $\varepsilon_0 > 0$ and $C > 0$, depending only on n and the geometry of N, such that the following holds.*

If $u \in W^{1,2}(B_r(x_0), N)$ is an energy-minimizing harmonic map *and*

$$r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \leq \varepsilon_0, \tag{5.4}$$

then u is smooth in $B_{r/2}(x_0)$ and satisfies the estimate

$$\sup_{B_{r/2}(x_0)} |\nabla u| \leq C\, r^{-1} \left(r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \right)^{1/2}. \tag{5.5}$$

5.2. **Estimate on the size of singular set and partial regularity.** Using the epsilon regularity result, we say that the energy minimizing harmonic maps are regular away from a measure 0 set. Indeed, one can estimate the Hausdorff dimension of the singular set.

Theorem 5.3 (Schoen–Uhlenbeck partial regularity)**.** *Let $n \geq 3$ and let N be a compact Riemannian manifold. Suppose that*

$$u \in W^{1,2}(\Omega, N)$$

is an energy-minimizing harmonic map. *Then there exists a closed set*

$$\Sigma \subset \Omega$$

such that

1. *$u \in C^\infty(\Omega \setminus \Sigma)$;*
2. *the singular set Σ has Hausdorff dimension at most $n-2$, i.e.*

$$\dim_{\mathcal{H}}(\Sigma) \leq n-2.$$

Proof. An important tool we use is the notion of energy density and the monotonicity formula.

For $x \in \Omega$ and $r > 0$ with $B_r(x) \subset \Omega$, define the scaled energy

$$\theta(x,r) := r^{2-n} \int_{B_r(x)} |\nabla u|^2.$$

Since u is an energy-minimizing (hence stationary) harmonic map, the monotonicity formula implies that $\theta(x,r)$ is nondecreasing in r. Therefore the limit

$$\theta(x,0^+) := \lim_{r \downarrow 0} \theta(x,r)$$

exists for every $x \in \Omega$.

Let $\varepsilon_0 > 0$ be the constant from the ε–regularity theorem and define the singular set

$$\Sigma := \left\{ x \in \Omega : \theta(x,0^+) \geq \varepsilon_0 \right\}.$$

By the ε–regularity theorem, u is smooth in $B_{r_0/2}(x_0)$. Hence

$$u \in C^\infty(\Omega \setminus \Sigma).$$

It is not hard to see that the regular set is open and the singular set is closed.

Now fix a compact set $\Omega' \Subset \Omega$. For each $x \in \Sigma \cap \Omega'$, by definition of Σ and monotonicity, there exists $r_x > 0$ such that $B_{5r_x}(x) \subset \Omega$ and

$$\int_{B_{r_x}(x)} |\nabla u|^2 \geq \varepsilon_0\, r_x^{n-2}. \tag{5.6}$$

The family $\{B_{r_x}(x)\}_{x \in \Sigma\cap\Omega'}$ is a covering of $\Sigma \cap \Omega'$. By the Vitali covering lemma, there exists a countable subcollection $\{B_{r_i}(x_i)\}$ such that

(1) the balls $B_{r_i}(x_i)$ are pairwise disjoint,
(2)
$$\Sigma \cap \Omega' \subset \bigcup_i B_{5r_i}(x_i).$$

So by (5.6) and the disjointness of the balls,

$$\varepsilon_0 \sum_i r_i^{n-2} \leq \sum_i \int_{B_{r_i}(x_i)} |\nabla u|^2 = \int_{\cup_i B_{r_i}(x_i)} |\nabla u|^2 \leq \int_\Omega |\nabla u|^2 < \infty.$$

Therefore

$$\sum_i r_i^{n-2} \leq \frac{1}{\varepsilon_0} \int_\Omega |\nabla u|^2.$$

Using the covering by $B_{5r_i}(x_i)$ and the definition of Hausdorff measure, we conclude

$$\mathcal{H}^{n-2}(\Sigma \cap \Omega') \leq C(n) \sum_i (5r_i)^{n-2} \leq \frac{C(n)}{\varepsilon_0} \int_\Omega |\nabla u|^2 < \infty.$$

Since $\Omega' \Subset \Omega$ was arbitrary, it follows that $\mathcal{H}^{n-2}(\Sigma)$ is locally finite in Ω, and hence

$$\dim_{\mathcal{H}}(\Sigma) \leq n - 2.$$

$\square$

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