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APPLICATIONS OF FLUID MECHANICS AND SIMILITUDE TO SCALE-UP PROBLEMS

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APPLICATIONS OF FLUID MECHANICS AND SIMILITUDE TO SCALE-UP PROBLEMS

PART I

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A brief review of the principles of dynamic similitude is given. Basic concepts of fluid mechanics are used to develop relations between fluid motion, equipment size, and fluid properties that may apply to chemical engineering work. A general method is given whereby the requirements for dynamic similitude for any flowing system can be determined. Examples are given showing how these principles can be used in pilot plant models to obtain scale-up data for operations involving resistance to fluid flow, discharge of liquids from tanks, blending of liquids by a mixer, control of forced vortexes, dissolving of solids during mixing, and absorption and desorption of gases in moving liquids. Suggestions are made for applying the principles to any type of operation involving mass transfer in a liquid, and to other flow operations such as fluidized systems and suspensions.

THIRTY-FIVE years ago Lord Rayleigh wrote (*12*), "I have often been impressed by the scanty attention paid even by original workers in physics to the great principle of similitude." This author feels impelled today to say, "I have often been impressed by the scanty attention paid by chemical engineers to the great principle of similitude."

Pilot plant studies are frequently initiated for the purpose of obtaining reaction rate and equipment performance data so that large scale equipment can be sized to give desired operational capacity. This discussion deals with processes wherein liquid motion is involved and where the reaction rate is controlled in full or in part by a concentration distribution in the liquid phase. If data obtained on a small-sized operation of this type are to be applicable to large scale design it is essential to know the effect of size on liquid motion and on the distribution of concentration. Unless the concentration pattern of the pilot can be achieved on the large scale operation, it will not be possible to duplicate the rate of action at the same temperature. And, unless the concentration pattern can be related to liquid motion, it will not be possible to predict the rate of reaction for different motion patterns.

It is intended here to call attention to principles of fluid mechanics and similitude which must be considered in the projection of experimental chemical engineering programs and for the evaluation of pilot plant data if such information is to become the basis for scale-up.

Some of the process operations involving liquid motion where size may have a significant effect on reaction rate are: mixing, liquid-liquid extraction, absorption, and desorption of gases in liquids, solution of solids, distillation, and heat transfer. In general, any mass- or heat-transfer operation involving a reaction at an interface is dependent on fluid motion and may therefore be dependent upon the size and shape of the system.

To illustrate the connection between fluid mechanics and process operations, the reader can consider, for example, the desorption of a gas from a liquid under such conditions that the rate is primarily a function of the rate at which the gas molecules move through the liquid to the liquid-gas interface. This is a case ordinarily referred to as one in which the "liquid film coefficient" is said to be controlling.

When the liquid with dissolved gas is in a quiet unagitated pool the rate at which gas molecules leave the interface and are desorbed depends upon the transport of molecules to the interface, and such transport is known as molecular diffusivity and natural convection. Diffusion coefficients for such operations have been evaluated for many substances. If forces are exerted on the liquid to produce fluid motion, then material will be transported to the interface by forced convection as well as by natural convection. When the liquid is agitated by a mixing impeller or by flowing through a bed of packing, the transport of molecules to the interface will be affected by the liquid motion. It would, therefore, be expected that the rate of desorption of the gas would be a function of the liquid motion as well as of its specific diffusivity. And, it follows, that evaluation of this effect can be made in terms of the liquid film coefficient of mass transfer, since the postulated "film theory" is convenient for handling transfer of material by convection. Accordingly, a liquid film coeffi-

TABLE 1. FORCE CHARACTERISTICS IN FORCE, LENGTH, TIME DIMENSIONS.

FLUID PROPERTY	SYMBOL & FORCE DIMENSION	KINEMATIC NOTATION force / mass	KINEMATIC DIMENSION	DIMENSIONLESS NUMBER NAME	GROUP	GROUP
F = Ma	$\rho = \frac{FT^2}{L^4}$	$a = \frac{F}{M}$	$\frac{L}{T^2}$	INERTIA	$\frac{FT^2}{\rho L^4}$	$\frac{F}{\rho u^2 L^2}$
SPECIFIC WEIGHT (gravity)	$\gamma = \frac{F}{L^3}$	$g = \frac{\gamma}{\rho}$	$\frac{L}{T^2}$	FROUDE	$\frac{L}{T^2 g}$	$\frac{u^2}{Lg}$
VISCOSITY	$\mu = \frac{FT}{L^2}$	$\nu = \frac{\mu}{\rho}$	$\frac{L^2}{T}$	REYNOLDS	$\frac{L^2}{T\nu}$	$\frac{Lu}{\nu}$
SURFACE TENSION	$\sigma = \frac{F}{L}$	$\omega = \frac{\sigma}{\rho}$	$\frac{L^3}{T^2}$	WEBER	$\frac{L^3}{T^2\omega}$	$\frac{Lu^2}{\omega}$
MODULUS OF ELASTICITY	$K_e = \frac{F}{L^2}$	$e = \frac{K}{\rho}$	$\frac{L^2}{T^2}$	CAUCHY	$\frac{L^2}{T^2 e}$	$\frac{u^2}{e}$

u = VELOCITY = L/T

cient is a function of fluid motion and therefore the same fluid motion would be necessary in two sizes of equipment in order to achieve the same film coefficient and the same rate of desorption.

If fluid motion is imposed on the liquid behind the reaction interface, there will result a definite pattern of flow and of concentration of molecules. Therefore, it is essential to understand the effect of all forces and boundary conditions on the liquid flow pattern, because these conditions will affect the distribution, or concentration, and the transport of the molecules.

It has been noted by Hutchinson and Sherwood (*9*) and by others, that liquid film coefficients in absorption processes are indeed dependent upon agitation and liquid motion. Other processes just referred to have also been observed to depend on agitation, but few attempts have so far been made to relate these phenomena in a comprehensive way to the mechanics of fluid motion (*15*). The application of the principles of similitude in fluid motion appear to have wide application to chemical processing. These principles have been used extensively in heat-transfer and certain mass-transfer operations (particularly in the gaseous phase), but little has been done to prove their value in planning small scale process experiments where liquid motion is important, and in interpreting results for the design of larger equipment to accomplish equivalent results. In the field of mixing, for example, many pilot plant studies have been completely useless for scale-up purposes because of neglect to consider some of the parameters of fluid motion.

If it is known that a chemical process rate is dependent on fluid motion behind an interface or in a flowing stream, it should then be possible to reproduce the same chemical rate in a small experimental model or in a large unit provided that dynamic similarity of flow can be achieved. The condition of dynamic similarity is necessary rather than geometric or kinematic similarity alone, because forces are required to move the mass of the substance in the manner desired. Unless dynamic similarity can be achieved or closely approximated, it will not be possible to reproduce results on any other scale if all other operating variables such as pressure and temperature are held constant.

Similitude and Forces

The forces and directions of flow at various points in liquids moving in mixing vessels, packed towers and other processing equipment are difficult to evaluate. Flow motions in process equipment are very complex, and point conditions can be estimated sometimes only by visual means. And when it is not possible to write a point-to-point description of the flow pattern and forces involved, recourse is found in the principles of similitude that have been developed in the field of fluid mechanics. These principles provide a means whereby the parameters which affect fluid motion can be used to define motion in a macroscopic sense without the necessity of a detailed knowledge of point conditions of flow.

The basis of the principles of similitude of fluid motion lie in the assumption that when a force is applied to a fluid mass, the resulting acceleration assumes a direction dependent upon the boundaries of the system and the physical properties of the fluid. Boundaries take into account the shape, size, and location of all components of the system (*3*). Physical properties include the specific weight, density, viscosity, surface tension, compressibility. Thus, if two systems of different size have the same ratio of length for all corresponding boundaries and positions, the systems are said to be geometrically similar. Fluid motions are kinematically similar when the paths are geometrically similar and the ratio of velocities at corresponding points are equal to the ratio at other corresponding points. Finally, a condition of dynamic similarity will exist when geometric and kinematic similarity are present and when the ratio of masses and forces at corresponding points are equal to the ratios at other corresponding points. It follows that pressure intensities must also bear the same relation as forces.

A fluid motion may be described completely in terms of force, mass, length, and time, because all boundaries, velocities and fluid properties can be put in these dimensions (*3*). It is convenient to use the three units of force, length, and time (F-L-T) and to relate force and mass by Newton's second law, $F = Ma$. It is obvious that all linear boundaries of a system can be measured by the length dimension L. Velocities of motion may be designated by length and time, or by u which is equal to L/T. Direction of velocity can be given in linear coordinates. Forces may be represented by Ma or by pressure intensity p, where $p = F/L^2$. The physical properties of fluids can also be represented in force, length, and time Specific weight γ is F/L^3, and thus gravity g is γ/p, or L/T^2. Density p (slugs per cubic foot) in force units is FT^2/L^4. Viscosity μ in force units is FT/L^2, and kinematic viscosity ν is μ/p, or L^2/T. Surface tension σ is F/L and kinematic interfacial tension ω is σ/p, or L^3/T^2. Elastic modulus c is F/L^2. All of the foregoing relations are listed for convenience in Table 1.

The kinematic notations are in terms of force per unit mass, and are therefore the ratio of the force property to the mass density.

The basic relations for similitude require equal ratios of dimensions, flows, and forces for corresponding points in two systems. Using subscripts 1 and 2 to denote two systems of different size, and subscript r for ratio, the following are the basic similitude equations:

Newton's second law:

$$F_r = M_r a_r \text{ and } F_r = \frac{\rho_r L_r^4}{T_r^2} \quad (1)$$

and

$$F_r = p_r u_r^2 L_r^2 \qquad (1a)$$

or, in terms of pressure intensity where $p = F/L^2$

$$p_r = \frac{\rho_r L_r^2}{T_r^2} \qquad (1b)$$

and

$$p_r = \rho_r u_r^2 \qquad (1c)$$

Also,

$$F_1 = F_r F_2 \qquad (2)$$

and

$$p_1 = p_r p_2 \qquad (2a)$$

and for dynamic similarity F_r or p_r must be constant.

For length, or coordinate position

$$L_1 = L_r L_2 \qquad (3)$$

L_r must be a constant for geometric similarity.

For time,

$$T_1 = T_r T_2 \qquad (4)$$

and T_r must be constant for kinematic similarity.

For the derived and physical properties:

for velocity, $u = L/T$:

$$u_1 = u_r u_2 \qquad (5)$$

and u_r must be a constant for kinematic similarity.

For density,

$$\rho_1 = \rho_r \rho_2 \qquad (6)$$

for viscosity,

$$\mu_1 = \mu_r \mu_2 \qquad (7)$$

or

$$\nu_1 = \nu_r \nu_2 \qquad (7a)$$

for specific weight

$$\gamma_1 = \gamma_r \gamma_2 \qquad (8)$$

or

$$g_1 = g_r g_2 \qquad (8a)$$

for interfacial tension

$$\sigma_1 = \sigma_r \sigma_2 \qquad (9)$$

or

$$\omega_1 = \omega_r \omega_2 \qquad (9a)$$

for elasticity

$$e_1 = e_r e_2 \qquad (10)$$

Force Ratios

The product of mass times acceleration is called an inertia force and is a resultant and equivalent to all forces acting on the mass. Hence, the inertia force can be equated to all forces entering into a fluid motion. It follows that the ratio of the inertia force to the other forces involved are dimensionless groups which can be used also to describe fluid motion. If such a ratio of inertia force to a fluid property force is constant for two sized geometrically similar systems wherein that property force is the only one operative then dynamic similarity obtains. This is *equivalent to saying that F_r remains* constant in Equation (2) when L_r is held constant in Equation (3).

It is possible to write by inspection the various ratios of the inertia force to other forces. Such ratios are known by name. For example, the ratio of inertia force to viscous force (*see* Table 1) is:

$$\left(\frac{\rho L^4}{T^2}\right) \div \left(\frac{\mu L^2}{T}\right) = \frac{L^2 \rho}{T \mu}$$

or

$$\frac{Lu\rho}{\mu} \text{ (Reynolds number)} \qquad (11)$$

Ratios for the other forces can be found in the same way and are listed in Table 1. They are:

$$\text{Froude number } \frac{u^2}{Lg} \qquad (11a)$$

$$\text{Weber number } \frac{u^2 L}{\omega} \qquad (11b)$$

$$\text{Cauchy number } \frac{u}{e^{1/2}} \qquad (11c)$$

If it can be experimentally determined that a fluid motion is dependent upon viscosity, then the value of the Reynolds number must be constant for any values of size L if the force ratio at these sizes is to be constant. Conversely, if it can be shown that the Reynolds number is constant at various sizes, then dynamic similarity will have been achieved if viscosity is the only active fluid property force.

Velocity Ratio in Scale-up

It is also evident that the inertia force can be equated to any one of the fluid property forces. This will result in a relation useful for scale-up to dynamically similar conditions. For example, considering again the case of the viscous *force; if it is the only fluid property* force effective, then

$$\frac{\rho L^4}{T^2} = \frac{\mu L^2}{T}$$

and by rearranging for velocity

$$\frac{L}{T} = \frac{\mu L^2 T}{\rho L^3 T} \text{ or } u = \frac{\mu}{\rho L}$$

and for similarity, applying Equations (3), (5), (6) and (7)

$$u_r = \frac{\nu_r}{L_r} \qquad (12)$$

This states that the velocity ratio at corresponding points in two systems is directly proportional to the kinematic viscosity and inversely proportional to the length ratio, or scale, of the two *systems. Therefore, under these force* conditions, velocity occurrences at cor-

TABLE 2. REQUIREMENTS FOR SIMILITUDE.

	RATIOS WHICH MUST REMAIN CONSTANT	DIMENSIONLESS GROUP	FOR SCALE-UP: VELOCITY RATIO REQUIRED	FOR SCALE-UP: WHEN WEIGHT AND VISCOSITY BOTH ACT	FOR SCALE-UP: WHEN WEIGHT AND TENSION BOTH ACT	FOR SCALE-UP: WHEN VISCOSITY AND TENSION BOTH ACT	EXAMPLE OF VELOCITY RATIO FOR SCALE-UP OF 2 SINGLE FORCE CONTROLLING
FORCE (Inertia)	$\frac{\rho_r L_r^4}{T_r^2} = Fr$	$N_I = \frac{F}{\rho u^2 L^2}$	$u_r = \sqrt{\frac{F_r}{\rho_r L_r^2}}$				$u_r = 0.5$
LENGTH	$\frac{L_1}{L_2} = L_r$	L_r	$L_r = C$	$L_r = \nu_r^{2/3}$	$L_r = \sqrt{\omega_r}$	$L_r = \frac{\nu_r^2}{\omega_r}$	
TIME	$\frac{T_1}{T_2} = T_r$	T_r	$T_r = C$				
VELOCITY	$\frac{u_1}{u_2} = u_r$	u_r					
DENSITY	$\frac{\rho_1}{\rho_2} = \rho_r$	ρ_r					
PRESSURE	$\frac{p_1}{p_2} = p_r$	$N_p = \frac{p}{\rho u^2}$	$u_r = \sqrt{\frac{p_r}{\rho_r}}$				
WEIGHT	$\frac{\gamma_1}{\gamma_2} = \gamma_3$	$N_{Fr} = \frac{u^2}{Lg}$	$u_r = \sqrt{L_r}$	$u_r = \sqrt{L_r}$	$u_r = \sqrt{L_r}$		$u_r = 1.41$
VISCOSITY	$\frac{\nu_1}{\nu_2} = \nu_r$	$N_{Re} = \frac{Lu}{\nu}$	$u_r = \frac{\nu_r}{L_r}$	$u_r = \frac{\nu_r}{L_r}$		$u_r = \frac{\nu_r}{L_r}$	$u_r = 0.5$
INTERFACIAL TENSION	$\frac{\omega_1}{\omega_2} = \omega_r$	$N_{We} = \frac{Lu^2}{\omega}$	$u_r = \sqrt{\frac{\omega_r}{L_r}}$		$u_r = \sqrt{\frac{\omega_r}{L_r}}$	$u_r = \sqrt{\frac{\omega_r}{L_r}}$	$u_r = 0.71$
ELASTICITY	$\frac{e_1}{e_2} = e_r$	$N_C = \frac{u^2}{e}$	$u_r = \sqrt{e}$				
TO DETERMINE		RATIO OF INERTIA TO OTHER PROPERTY FORCE	EQUATE INERTIA TO OTHER PROPERTY FORCE	EQUATE VELOCITY RATIOS	EQUATE VELOCITY RATIOS	EQUATE VELOCITY RATIOS	SUBSTITUTE IN VELOCITY EQUATION

responding points in a system twice the size which contained the same fluid at the same temperature and pressure, would be one half of the velocities in the smaller system.

Equations like (12) for other properties are determined in the same way, they are:

When weight (or gravity) controls $u_r = (L_r/g_r)^{1/2}$, the force of gravity is approximately constant all over the surface of the earth and the ratio $g_r = 1$, so the relation is usually written as

$$u_r = \sqrt{L_r} \qquad (12a)$$

When interfacial tension controls

$$u_r = \sqrt{\frac{\omega_r}{L_r}} \qquad (12b)$$

When elasticity (or compressibility) controls

$$u_r = \sqrt{e_r} \qquad (12c)$$

These are listed in Table 2. They are referred to as velocity ratio in scale-up, or as "model laws."

By means of these relations it is possible to account for fluid motion differences or similarities in equipment of different size. It is also possible to compute equipment size necessary for desired fluid velocities.

Two Property Forces

When two fluid properties influence fluid motion a summation of the individual effect of each can be made, but without experimental data it is not possible to predict the relative effect of each. However, when two property forces are known to apply, each ratio of inertia force to property force must be constant for any change in the force, length, time scales. For example, assuming a fluid motion where both viscosity and specific weight are effective, the force ratios which can be used as parameters (Table 1) are the Reynolds number and the Froude number. From the previous discussion it is evident that for a different size system, geometrically similar, the Reynolds number must be constant to achieve dynamic similarity. Likewise, the Froude number gives the value of the ratio of inertia forces to weight forces (since $\gamma = \rho g$, the Froude number is often referred to as the ratio of inertia to gravity forces) and if F_r (weight) of Equation (2) is to be constant while L_r of Equation (3) is constant as required for similitude, then the Froude number also must be constant as size is changed.

These requirements for similitude impose restrictions on the scale-up possibilities. From Table 2 and Equation (12) the velocity ratio due to viscous forces in size change, or scale-up, is inversely proportional to size. So, using the same kinematic viscosity, if the L_r were 2, then the velocities in the larger system would be one-half those in the smaller. The velocity ratio in scale-up due to weight (or gravity) is given in Equation (12*a*) and in Table 2. Accordingly, for L_r of 2, the velocities in the larger system would be $\sqrt{2}$, or 1.41 times those in the smaller. The viscous forces thus tend to reduce velocity as size increases, whereas the weight forces increase velocities. For dynamic similarity to exist u_r must be constant for any size system regardless of what forces are operative. Therefore, u_r (viscosity) must equal u_r (weight). By equating the velocity scale ratios (Equations (12) and (12*a*))

$$\frac{\nu_r}{L_r} = \sqrt{L_r}$$

$$\nu_r = L_r^{3/2} \qquad (13)$$

This shows that dynamic similarity under these conditions cannot exist for the same fluid at the same temperature and pressure. If L_r is 2, then the kinematic viscosity of fluid in the large system must be $2^{3/2}$, or 3.83, times that in the small one.

Equation (13) states the relation between viscosity and scale for dynamic similarity of flow. To determine the velocities of flow at various scales, either of the two velocity ratio equations for scale-up will yield the same result, because the value of ν in Equation (12) is fixed by Equation (13) so that u_r (viscosity) will be equal to u_r (weight).

Other pairs of force properties can be handled in the same way. Several possible combinations are listed in Table 2. They are:

For the case when specific weight and interfacial tension play the significant roles, the velocity ratios for each are equated, resulting in

$$L_r = \sqrt{\omega_r} \qquad (13a)$$

When viscosity and interfacial tension are both effective,

$$L_r = \frac{\nu_r^2}{\omega_r} \qquad (13b)$$

The relations which have been developed in the foregoing discussion are summarized in Tables 1 and 2. The various force dimensions, notations and dimensionless groups are given in Table 1. Table 2 shows the equations required for similitude for various fluid property forces, normally affecting fluid motion.

Distorted Models

It may not be convenient or possible to change such properties as interfacial tension and viscosity or other properties as required by the relations of Table 2 to achieve similarity. In such cases the effect of one or more property may be minimized by distorting the model. For example, interfacial tension is minimized by avoiding sharp fluid surface curvature.

If viscosity is one of two effective forces it is possible to minimize its effect, or even to render it negligible by operating at high velocities to achieve high Reynolds numbers. If at the same time it is possible to evaluate the other force parameter, it may then be possible to approximate a combined effect. In the event that individual parameter effects can be evaluated by such means it can be assumed for practical purposes that the individual forces dependent upon each parameter can be added together. Such is the technique as developed by Froude for the case of viscous and gravity forces operating on the hull of a ship, and the use of models for experimental work to determine the power necessary to drive ships of various designs is based on the principle that the force functions are additive. This is analogous to the well-known experimental technique of holding all variables constant except one, so that each variable may in turn be evaluated.

More than Two Property Forces

When three force properties play a part in fluid motion it is rarely possible to reproduce dynamic similarity on any other size equipment. It is clear from inspection of the velocity ratios for scale-up that if more than two of them are equated at a time, it restricts the physical properties of the fluids so that there is little possibility of choosing actual liquids that will meet the requirements. It is sometimes possible to distrot models (*5*) and to evaluate the effect of individual forces independently: this was discussed in the preceding paragraph.

Determination of Applicable Parameters

Equations (1)-(10) can be used to describe all flow motions, in which they represent all the variables and boundaries. However, the interrelation between them must be known and related to Newton's second law as shown in Equations (10)-(13) and as tabulated in Table 2. The interplay of forces due to fluid properties, boundary conditions, and any imposed work force results in a definite flow regime. Such an interplay of forces may result in various combinations of the similitude equations. If an equation can be written to describe a fluid motion completely, then a grouping of properties, boundaries, and forces can be found which can be used as a parameter for conditions of dynamic similarity. Unfortunately such a technique is often difficult and, therefore, cut-and-trial means for determining parameters are frequently used. The relations just reviewed cannot be used to predict which forces will take part or to what degree they might interact, except in a qualitative way.

Inertia forces predominate whenever the flow pattern is constant over a relatively wide range of velocities. Inertia forces are, of course, always present in any flow motion, as postulated by Newton. Since the property force parameters are all ratios of inertia to the property force, it follows that for high values of the Reynolds, Froude, and Weber numbers, the inertia forces predominate, and viscous, weight, and interfacial tension forces play a minor role. Conversely, low values of these parameters indicate that the corresponding fluid properties may play an effective role.

Weight (or gravity) forces are effective wherever waves or vortexes are present in a fluid interface, and when the Froude number is relatively small.

Viscous forces are effective when the viscosity is high, or when there is not much turbulence, or when the Reynolds number is low.

Interfacial tension forces may be important when there is sharp surface or interface curvature or where interfacial tensions are large, or where the Weber number is relatively low.

Elastic forces may be of importance when the modulus of elasticity or compressibility is small, or where pulsations are frequent, or when the Cauchy number is high.

One technique often used to determine which parameters may be useful in correlating or characterizing flow motion is that of Dimensional Analysis. The Pi theorem of Buckingham (*3*), and the method of Rayleigh (*12*) are based on the premise that a general function of all variables which contribute to a flow motion can be stated, and that these variables can be arranged in a variety of dimensionless groups depending on the arbitrary choice of four variables. Either of these two methods can be used and will give identical results. The Froude, Reynolds, Weber, and Cauchy numbers can all be derived in identical form to those just shown and in Table 2. In addition, all the length ratios which are inherent in the boundary dimensions of the system can be derived.

To illustrate the use of these techniques, assume that all factors influencing a flow motion are: (1) A series of linear (L) boundary dimensions a, b, and c. (2). A mean velocity u in the direction of a pressure intensity p. (3). Specific weight γ, density ρ, and viscosity μ of the fluid.

Then

$$f(a,b,c,u,p,\gamma,\rho,\mu) = 0 \qquad (14)$$

If this is operated on by either the Buckingham or Rayleigh technique (*18*), the result is:

$$\frac{p}{\rho u^2} = f\left(\frac{a}{b}, \frac{a}{c}, \frac{u^2}{ag}, \frac{au}{\nu}\right) \qquad (15)$$

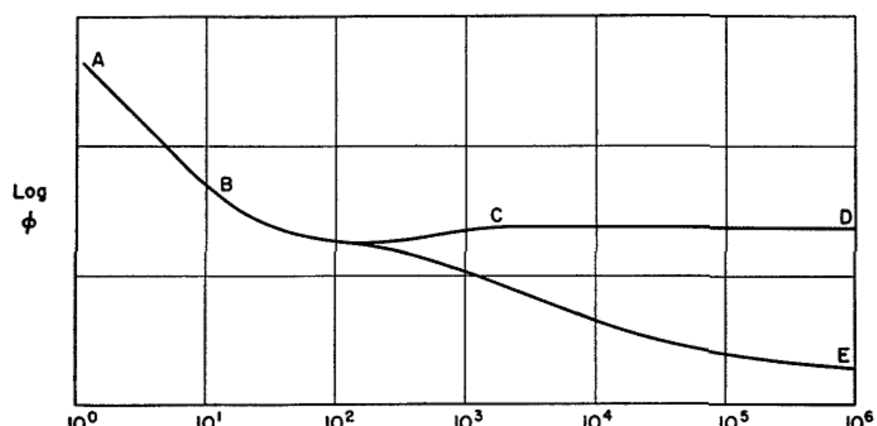
The left-hand term will be recognized as the Newton inertia force group in pressure units (Eq. 1*c*). The first and second terms on the right are both linear ratios, and the third and fourth terms are the Froude and Reynolds numbers respectively. The a dimension was chosen as the reference length. Note that the dimensionless ratios of inertia force to property force as derived earlier from Equation (1) could have been written directly in the form of Equation (15), and the boundary dimension ratios are a requirement of geometric similarity since all dimension ratios must stay constant for geometric similarity (Eq. (3)) if any one dimension is changed. So, if Equation (14) can be deduced, Equation (15) may be written by inspection based on the similarity requirements of Equations (1)-(10).

Regardless of the technique used to determine an equation like (15), such an equation does not of itself show the interrelation between the various factors involved. Furthermore, its validity is dependent upon that of Equation (14). Hence, the importance and interrelation of the groups in an equation like (15) are not known until they are evaluated by experiment. If experimental data are available then each dimensionless ratio of an equation like (15) can be used in turn as a parameter and evaluated. Graphical means are often employed for such evaluations. Figure 1 is one example showing how the inertia forces are evaluated in terms of the Reynolds and Froude numbers for fluid motion in a mixing tank (*18*). At best, however, these methods simply show the effect of various properties and geometry by the indirect method of assuming an influence and then substantiating it if possible by experiment: a type of cut-and-trial technique to obtain parameters for correlation of data.

There is a direct method by which Equations (1)-(10) can be used to give the significant parameters for flow motion directly. It is dependent upon ability to write a differential equation to express a particular flow motion. The technique can best be illustrated by the general equations of Navier-Stokes for the case of noncompressible viscous flow (*21*). The first of the Navier-Stokes equations for flow, omitting the divergence term, is for system 1.

$$-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x_1} + \frac{\mu_1}{\rho_1}\left(\frac{\partial^2 u_1}{\partial x_1{}^2} + \frac{\partial^2 u_1}{\partial y_1{}^2} + \frac{\partial^2 u_1}{\partial z_1{}^2}\right) = \frac{u_1 \partial u_1}{\partial x_1} + \frac{v_1 \partial u_1}{\partial y_1} + \frac{w_1 \partial u_1}{\partial z_1} + \frac{\partial u_1}{\partial T_1} \qquad (16)$$

If this equation is written for another system 2, with the proper alteration of length, time, and force (or pressure in this case) scales, the result will be an equation for a completely similar flow. The proper scales for similarity will be those of Equations (1)-(10) which may apply. Thus if Equation (3) in terms of x is written as $x_1 = L_r x_2$, and similarly for y and z, and also Equations 2*a*,



$\text{Log}\ \frac{Lu}{\nu}$, or $\text{Log}\ \frac{D^2 N \rho}{\mu}$

FOR CURVE -ABCD- NO VORTEX PRESENT $\phi = \begin{cases} N_I = \frac{FT^2}{\rho L^4} \\ \text{or } N_P = \frac{Pg}{\rho N^3 D^5} \end{cases}$

FOR CURVE -BE- VORTEX PRESENT $\phi = (N_I)(N_{Fr})^{-n}$

Fig. 1. Power characteristics of a mixing impeller.

4, 5, 6, and 7 are substituted in Equation (16) the result is

$$-\frac{p_r}{\rho_r L_r}\left(\frac{1}{\rho_2}\right)\left(\frac{\partial p_2}{\partial x_2}\right)$$

$$+\frac{\mu_r u_r}{\rho_r L_r^2}\left(\frac{\mu_2}{\rho_2}\right)\left(\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial y_2^2}+\frac{\partial^2 u_2}{\partial z_2^2}\right)$$

$$=\left(\frac{u_r^2}{L_r}\right)\left(\frac{u_2\partial u_2}{\partial x_2}+\frac{v_2\partial u_2}{\partial y_2}+\frac{w_2\partial u_2}{\partial z_2}\right)$$

$$+\frac{u_r}{T_r}\left(\frac{\partial u_2}{\partial T_2}\right) \quad (17)$$

For dynamic similarity Equation (17) must equal Equation (16). This is true when the coefficients of Equation (17) are equal, or

$$\frac{p_r}{\rho_r L_r}=\frac{\mu_r u_r}{\rho_r L_r^2}=\frac{u_r^2}{L_r}=\frac{u_r}{T_r} \quad (18)$$

These groups each must be satisfied for dynamic similarity for viscous flow.

Inspection of the third and fourth terms show that they are obviously equal.

The second and third terms,

$$\frac{\mu_r u_r}{\rho_r L_r^2}=\frac{u_r^2}{L_r}$$

$$\frac{L_r u_r \mu_r}{L_r^2 u_r^2 \rho_r}=1 \quad \text{or} \quad \frac{\mu_r}{L_r u_r \rho_r}=1$$

and is dimensionless, and by substituting according to Equations (3), (5)-(7) there results

$$\frac{L_1 u_1 \rho_1}{\mu_1}=\frac{L_2 u_2 \rho_2}{\mu_2}=N_{Re} \quad (19)$$

The Reynolds number so derived shows that this form is necessary for dynamic similarity.

Equating the first and third terms of Equation (18) gives

$$\frac{p_r}{\rho_r L_r}=\frac{u_r^2}{L_r}$$

or $p_r = \rho_r u_r^2$ which is Equation (1*c*). Equation (1*c*) is equal to Equation (1), and it may be written as

$$\frac{F_r}{\rho_r u_r^2 L_r^2}=1 \text{ the Newton inertia group} \quad (20)$$

substituting, and letting L^2 = area A

$$\frac{F_1}{\rho_1 u_1^2 A_1}=\frac{F_2}{\rho_2 u_2^2 A_2}=N_I$$

This is sometimes called the drag coefficient, and is the other dimensionless parameter which must be constant in dynamically similar viscous flow.

Equating the first and second terms of Equation (18)

$$\frac{p_r}{\rho_r L_r}=\frac{\mu_r u_r}{\rho_r L_r^2} \text{ and let } p = F/L^2$$

then

$$\frac{F_r \rho_r L_r^2}{L_r^2 \rho_r L_r \mu_r u_r}=\frac{F_r}{L_r \mu_r u_r}=1 \quad (21)$$

and this is another dimensionless form of the Newton inertia group. Either Equation (20) or (21) can therefore be used to characterize the inertia forces.

Thus, from the differential equation it is shown that the Reynolds and inertia groups are sufficient to satisfy dynamic similarity for viscous flow. If, therefore, a differential equation can be written to describe any complex flow motion, the dimensionless parameters necessary for similarity can be obtained directly, and those obtained in this way should be complete and sufficient for the purpose. If the differential equation can be solved, the interrelation between the parameters can be obtained. Otherwise the interrelations must be obtained through experimental data.

Experimental Data

Experimental data involving force and various values of length and time can be used as shown in Figure 1 to relate the various fluid motion parameters (*18*). Figure 1 is a curve relating the inertia force to the Reynolds and Froude numbers for a mixing impeller, but the curve *ABCD* is also typical of the general relation for fluid drag in a pipe or across any solid surface. A brief review of the significance of this logarithmic plot is pertinent.

At low Reynolds numbers the part of the curve *AB* has a slope of −1 and can be represented (from Equation (15)) by

$$\frac{F}{\rho u^2 L^2}=K'\left(\frac{Lu\rho}{\mu}\right)^{-1}$$

giving

$$F = K'Lu\mu \quad (22)$$

showing that force is proportional to size, velocity and viscosity under these conditions, called viscous flow, and that these relations hold in similitude. The value of K' is determined from the position of the curve, and is thus fixed by experimental data.

For that part of the curve *CD*, the slope is zero and

$$\frac{F}{\rho u^2 L^2}=K''\left(\frac{Lup}{\mu}\right)^{\circ}$$

giving

$$F = K''\rho u^2 L^2 \quad (23)$$

This shows that viscous forces are not controlling under these conditions. The flow is one of fully developed turbulence. The value of K'' can be determined from the curve and is fixed by experimental data.

For curves such as *BE*, similar equations can be developed depending on the slope, but in these cases it is evident that viscous forces are effective over the entire curve *ABE*.

Correlation of experimental data in this way allows evaluation of the importance of the force parameters. It is evident that the slope of the line in such a plot as in Figure 1 gives the relative effect of the inertia group and the Reynolds group, and thus the interaction of properties in geometrically similar systems. When the slope of the line *AB* is −1 it shows that the inertia force is directly proportional to the viscous force: hence, viscous forces alone are dominant in defining the flow motion. When the slope of the curve is zero then the inertia force is independent of viscous forces. Furthermore, since similitude is based on Newton's law, the flow pattern is defined by the inertia group. Hence, a flow pattern or motion pattern will be the same in two different sized systems only when the inertia group is the same. If viscous forces control, as for *AB*, then they affect the flow pattern: a change of Reynolds number will change the flow pattern. Therefore, while it is possible to reproduce the same flow pattern in two different sized systems it is not possible to reproduce the same velocities in the two systems with the same viscosity liquid. This is a corollary of Equation (12).

When viscous forces do not control line *CD*, then the Reynolds number has no effect on the flow pattern, and the flow pattern is constant over the entire range of fluid velocities represented by all the Reynolds numbers from *C* to *D*. A velocity at one point in a system will bear a definite relation to all other velocities in the same system, and in a kinematically similar system, the velocity distribution will have the same relations at corresponding points. Therefore, if the inertia group is constant, the velocity at one point (or average) can be varied and the same flow motion pattern will result. It is possible, then, to reproduce a flow motion pattern and the same velocities of flow at corresponding points in systems of two different sizes when there is fully developed turbulence. Fully developed turbulence may be defined by a slope of zero for the line *CD*.

The importance and interaction of other property forces can be analyzed in the same way provided that the data are available and sufficient to determine the effect of one force parameter independent of the others.

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APPLICATIONS OF FLUID MECHANICS AND SIMILITUDE TO SCALE-UP PROBLEMS

PART II

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A brief review of the principles of dynamic similitude is given. Basic concepts of fluid mechanics are used to develop relations between fluid motion, equipment size, and fluid properties that may apply to chemical engineering work. A general method is given whereby the requirements for dynamic similitude for any flowing system can be determined. Examples are given showing how these principles can be used in pilot plant models to obtain scale-up data for operations involving resistance to fluid flow, discharge of liquids from tanks, blending of liquids by a mixer, control of forced vortexes, dissolving of solids during mixing, and absorption and desorption of gases in moving liquids. Suggestions are made for applying the principles to any type of operation involving mass transfer in a liquid, and to other flow operations such as fluidized systems and suspensions.

Applications

Pilot plant operations may be thought of as of two types; those involving the kinematic properties such as viscosity and weight, and those which in addition to the kinematic properties are affected by vapor pressure, solubility, equilibrium, conductivity, and the like. The first type deals with fluid mechanics only, and can be illustrated by examples of fluid flow and certain mixing operations. The second type deals also with chemical reactions and concentrations, and includes such operations as gas absorption and extraction.

Typically Fluid Mechanical. The resistance of flow of fluids through equipment such as pipe lines and heat exchanges can be determined by model studies. The familiar Fanning friction factor and other similar relations are of common knowledge. In these systems it has been shown that the inertia group and the Reynolds number are sufficient to define dynamic similarity when all linear ratios are constant. Therefore, the resistance to flow measured in a model can be projected to the resistance in any larger geometrically similar system, and pumps and pipelines can be sized with accuracy. In such systems where there is no free liquid surface and where the fluid is continuous and incompressible, the Froude, Weber, and Cauchy parameters do not apply, and the Reynolds number and the inertia group alone control. A liquid of any viscosity and density can be used in the model to determine the relationship of the inertia group to the Reynolds number, and the results will apply to any other liquid in scale-up by using Equations (1)-(12).

The blending of miscible liquids is a mixing operation often carried out under conditions represented by curve *ABCD* of Figure 1. Normally the operations are carried out under conditions equivalent to the *CD* part of the curve, using side-entering propellers, properly off-centered, or other types of impellers using baffles. Viscous forces do not exert an influence. Under these conditions there will not be a vortex in the surface of the liquid, and weight or gravity forces will not play a part. A pilot plant model can, therefore, be constructed, and the time and power necessary to achieve a desired blend for a given set of feed conditions can be found. These conditions can be projected to a large size tank by keeping the inertia group constant, geometric similarity and equal liquid flow velocities.

The following example is taken from actual operating data in small and large tanks: A 6-in. diam. square pitch three-blade marine-type propeller operating in a side-entering properly off-centered position in a 15-ft. diam. vertical axis cylindrical tank, was found to blend two gasolines in 40 min. when running at a speed of 1680 rev./min. What size mixer and power are required to blend the two gasolines in the same length time in a 60-ft. diam. tank?

The scale-up will be made on the basis of equal inertia groups, geometric similarity, and equal flow velocities at corresponding points.

All linear dimensions will increase by the ratio of tank size or $60/15 = 4$. Liquid depth will be four times that in the smaller tank: the propeller will be a geometrically similar one 2 ft. in diameter. Liquids are the same. Convert force to power P by $FL/T = P$, then the inertia group may be written $PT^3/\rho L^5$, and

$$\frac{P_1 T_1^3}{\rho_1 L_1^5} = \frac{P_2 T_2^3}{\rho_2 L_2^5}$$

$$\frac{P_2}{P_1} = \left(\frac{T_1}{T_2}\right)^3 \left(\frac{L_2}{L_1}\right)^5$$

since time occurrences (and velocities) are to be constant $T_1 = 0.25 T_2$ (see line one, Table 2)

then

$$\frac{P_2}{P_1} = (0.25)^3 (4)^5$$

or

$$P_2 = 16\ P_1$$

Actually, the power required by the model is 0.768 hp., thus the power required by the 24-in. propeller is 16 (0.768) = 12.5 hp. The rotational speed of the 24-in. propeller is determined from propeller characteristics and is 420 rev./min. for 12.5 hp. Operating results confirmed the scale-up.

A different speed could, of course, be used and the same flow pattern would result, but the blending time would vary. It would then be necessary to know the relation between blending time and speed of rotation from the model, and scale-up made for the different speed condition.

It has been shown elsewhere (*16*) that when a forced vortex is formed by means of a rotating mixing impeller, a curve like *ABE* (Fig. 1), will result, and at high Reynolds numbers the Froude number is important. To reproduce a vortex geometrically similar in another size vessel, Equation (13) must be satisfied. The same liquid cannot be used in another sized vessel. Equation (13) requires that as size is increased the kinematic viscosity of the liquid must be greater if a similar vortex is to be formed. Figure 2 illustrates this case and shows a cross section of forced vortexes under conditions where similarity is and is not present in systems of two different sizes.

When liquids are withdrawn from tanks through pipes in the bottom, a vortex will form due to weight (or gravity) and viscous forces. The following example will show the use of the similitude relations to set up a pilot model for the case of removing liquid from a storage tank.

A 90-ft. diam. light oil storage tank is to be built and a draw-off pipe 18 in. in diameter is to be placed in the bottom of the tank, 6 ft. from the side wall, and extending vertically from the tank bottom to a height of 18 in. When oil is withdrawn and as the liquid level drops, a vortex will form above the outlet. Both the Reynolds and Froude parameters will be necessary to characterize the flow. If the vortex is allowed to extend to the bottom of the outlet line, air will be drawn into the oil line, and this is to be avoided. At what level of liquid in the tank should the flow be stopped to avoid air entrainment for a given draw-off rate? A small model can be built to determine this for the large tank. The model requirements are determiend as follows:

A piece of 0.45-in. I.D. tubing is available for a draw-off tube. Since the tank diameter is to be 90/1.5, or 60 times the draw-off tube size, the model should be 60(0.45), or 27 in. in diameter, and its height arranged accordingly. The scale of the large tank is 18/0.45 = 40 times that of the model. The draw-off tube should be placed 6/40, or 0.15 ft. from the side of the model, and the opening should be 0.45 in. above the bottom of the model. Since both the Reynolds and Froude numbers must remain constant over this scale ratio of 40, it is necessary that Equation (13) must apply. Therefore, it is evident that the oil to be used in the large tank cannot be used in the model. Rather, a much less viscous oil must be used in the model.

From Equation (13) $\nu_r = 40^{3/2} = 305$. Thus a liquid whose kinematic viscosity is $1/305$ times that to be handled in the large tank should be used in the model. With these dimensions and liquid properties, data from the pilot model can be scaled up by either the Reynolds or Froude number.

Assuming that the large tank is to have a drawn-off rate of 2400 gal./min., the quantity of discharge is equal to the velocity times area, or $Q = uL^2$.

From the Froude scale-up relation (Table 2)

$$u_r = \sqrt{L_r,}$$

and then

$$Q_r = u_r L_r^2,\ Q_r = L_r^{5/2} \qquad (24)$$

Thus

$$Q = 40^{5/2} = 10{,}000$$

and

$$Q_2 = \frac{2400}{10{,}000} = 0.24 \text{ gal./min.},$$

the discharge rate to be used in the model. If, at this rate, air is drawn into the outlet when the liquid level at a given point at the model wall is 0.8 in. above the bottom, then the level of the liquid at a corresponding position in the large tank must be 0.8(40), or 32 in. Or, this level would be 32-18, or 14 in. above the discharge opening. If the same liquid were used in the model as to be used in the large tank, the level at which air would be induced would be lower, hence when scaled up to the large tank the level would be too low and air would be induced at a higher level than supposed. It should be noted, however, that since the ratio of tank diameter to draw off tube diameter is large (60 to 1) any effect of a small change in this ratio should not be of great importance, hence the data might reasonably be applied to tanks of say 75 to 100 ft. in diameter. Similar reasoning would allow small variations in the model tank diameter.

Operations Involving Mass Transfer

Many absorption, desorption, extraction and solubility operations are affected by agitation—or fluid motion. There are many references to the beneficial effect of mixing or other methods of agitation on such operations, but few data can be found where the effect of size and similarity have been studied.

One study of gas absorption reported by Hutchinson and Sherwood (*9*) shows the effect of agitation of the liquid phase behind the liquid-gas interface, for a number of gases. Only one small size vessel and mixing impeller was used, but data were obtained for a change in impeller speed. A logarithmic plot of the absorption coefficient and impeller speed showed that the coefficient was constant at very low speeds, but that a constant positive slope was obtained when speed was increased in the higher speed ranges. It is evident that normal rates of diffusion were controlling the coefficient in the liquid at low stirring speeds, but that at higher speeds the forced fluid motion caused distribution of the solute at a greater rate than diffusion. If the same fluid motion could be achieved in a vessel of larger size, the data should be capable of reproduction in this larger system. It is inferred from a description of the apparatus that there may have been swirl and vortex in the liquid, and if this were true, then it is not possible to use these data for scale-up because of Equation (13). This work was undertaken to show the effect of liquid motion on a gas-absorption coefficient, and was not intended for scale-up data. It is referred to here to show that the effect of fluid motion has been shown by them to be dominant and controlling for absorptions when agitation is used.

The process of gas desorption is currently being studied in our laboratories under conditions where fluid motion can be controlled, and for the purpose of evaluating the relations which must be known for scale-up. Some data are available for desorption of carbon dioxide from the surface of water to air in a 12-in. and an 18-in. diam. cylindrical tank (*11, 13*). The liquid is agitated by a flat-blade turbine mixing impeller. Baffles are used in the tanks for some runs, while in others, the baffles have been omitted. Figure 3, curve *A* illustrates the data and correlation for the case of desorption with baffles in the two sized tanks. Data for operation without baffles are also plotted in Figure 3, curves *B* and *C*.

When baffles are used the flow is characterized by the curve *CD* of Figure 1, and the liquid surface, in contact with carbon dioxide free air is substantially level, so that it is clear that inertia forces control the operation and the Froude group does not apply. Accordingly, the same flow motion can be achieved in both sized tanks, both as regards pattern and velocity. The dimensionless group by which a mass-transfer coefficient k can be correlated is kL/d where, d is the diffusivity. This is plotted logarithmically against the Reynolds number Lu/ν. Data for both sized systems fall on a single line *A*, hence the relation of fluid motion and size are known for this reaction for a size ratio of 1.5 to 1. Also, the data show that dynamically similar fluid motion can be achieved at the two sizes and that this can be correlated with the desorption coefficient.

When baffles were removed from the two mixing tanks, vortexes were formed in each liquid, and it is apparent that

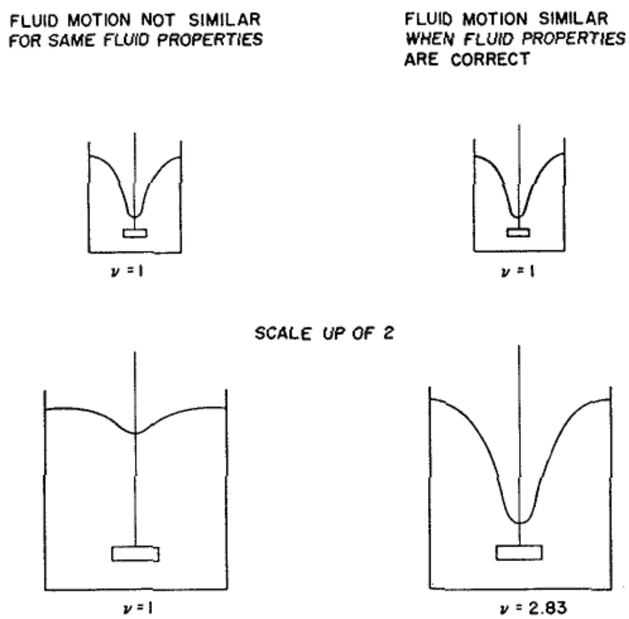

Fig. 2. Dimensionally similar systems, Reynolds and Froude forces effective.

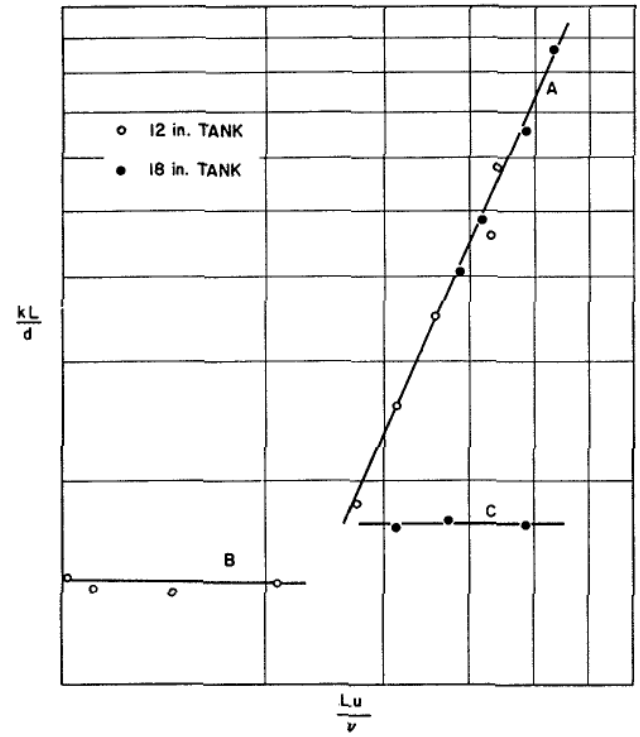

Fig. 3. Relation between transfer coefficient and Reynolds number.

the flow can be characterized by a line like *BE* (Fig. 1). Also, Figure 2, left-hand side shows the type of vortex formed for equal Reynolds numbers in each tank using the same liquid at the same temperature. Under these conditions, both viscous and weight forces are effective, and therefore, the conditions of Equation (13) must be satisfied if similar flow is to be achieved. The curves *B* and *C* show data for the small and larger tank respectively. The data are not correlated for size by the plot, and this is to be expected when similitude is not present. Data are now being obtained for dynamically similar conditions by lowering the temperature of the solution (thus increasing its kinematic viscosity to 1.84 times that in the smaller tank), and taking into account the different diffusivity occasioned by the lower temperature. It follows that desorption or absorption data obtained with appreciable vortexing of liquid cannot be duplicated in a different sized dynamically similar system, since it is not possible to do so with the same liquid and temperature.

Unfortunately, most chemical reaction rates and equilibria are sensitive to temperature, and pilot operations are run at temperatures giving optimum yields. If the temperature in the model experiments must be the same as in the large scale unit, then vortexes in the liquid must be avoided. In fact, the fluid motion must, in these cases, be such as to be controlled by not more than one fluid property force. Under baffled conditions in a mixing tank it is possible to use the same liquid at the same temperature in both the model and the prototype. These experiments show that the use of baffles in this desorption operation not only provide a better flow pattern for mixing (as evidenced by higher coefficients for equal power input) but also provides the necessary requirements for scale-up.

A number of references could be cited for data on the rate of solution of solid benzoic acid and solid salts in water, where the action took place in smooth wall cylindrical vessels without baffles using rotating impellers and deep vortexes were present. Such data correlate like those of lines *B* and *C* of Figure 3 and are useless for scale-up. Dynamically similar conditions on a larger scale cannot be achieved at the same temperature, and insufficient data were obtained at different liquid viscosities. Furthermore, complete geometric similarity of solid particle size was not achieved.

Some recent work of Mack and Marriner (*10*) on the rate of solution of solid benzoic acid in water were obtained in cylindrical vessels using rotating impellers and baffles. The conditions were correct for scale-up data, since no vortexes were present in the liquid surface and the flow characteristics were like curve *CD* of Figure 1. The data cover two sizes of tank and different solid "pill" size. When the data for the two size systems (pill ratio approximately constant) are recalculated to the mass-transfer group and plotted against Reynolds number (Fig. 3) the data plot in a straight line similar to Curve *A*. Thus, these data were obtained under such conditions that scale-up can be made—at least over the size range studied, a scale ratio of 2 to 1.

These examples illustrate that data necessary for scale-up for this type operation can be correlated, provided that the requirements of similitude are taken into account.

Heat Transfer

The techniques of correlation of convection heat-transfer data with fluid mechanics relations and similitude are well known. Heat transfer in mixing tanks has also been correlated by the conventional techniques, and a few of the data of Chilton, Drew and Jebens (*4*) show that data taken under conditions of only slight vortexing can be correlated by the Nusselt group or "*J* factor" and the Reynolds number. For two sizes of equipment, scale 4 to 1, the data correlated as shown by curve *A* of Figure 3, where the *J* factor replaces the ordinate in the figure.

Other data from our laboratories (*6, 19, 20*) for heat transfer in mixing tanks from 1 to 4 ft. in diameter using vertical tubes for heat transfer and baffles (and the absence of vortexes) correlate in the same way as curve *A*, Figure 3, using appropriate heat-transfer parameters. Since the data were taken under flow conditions which allowed scale-up and similitude simultaneously, they are use-

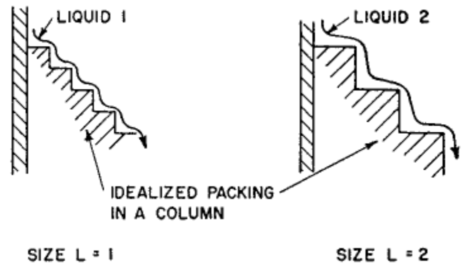

Fig. 4. Packed absorption column.

IF BOTH REYNOLDS AND FROUDE FORCES ARE EFFECTIVE:
TO ACHIEVE SIMILAR FLUID MOTION IN THE LIQUID THE KINEMATIC VISCOSITY OF LIQUID 2 MUST BE 2.83 TIMES THAT OF LIQUID 1.
IF THE SAME LIQUID VISCOSITY IS USED IN 1 AND 2 THE FLOW MOTIONS CANNOT BE SIMILAR, AND THE LIQUID FILM TRANSFER COEFFICIENTS CANNOT BE EQUAL.

ful for large size design. Successful scale-up of these data has been made to tanks 10 and 12 ft. in diameter (*17*).

Other Possible Applications

There are numerous data in the literature on the performance of packed towers and other equipment for gas-liquid and for liquid-liquid mass-transfer operations. Such operations involve fluid motion and are subject to the various kinematic forces. Few experiments have been made with the appreciation of the interplay of the various kinematic parameters in mind, and the fact that a change of scale and packing size may be of great significance in causing a change in fluid motion—all other things constant. Since at least two forces are usually operative in fluid motion in a packed column, it will be necessary to experiment in the pilot plant under "distorted" conditions of size or flow, so that the effective parameters can be evaluated individually. Figure 4 will serve to illustrate one possible situation in a packed absorption column where liquid is shown flowing down across steps of packing. It is clear that weight (or gravity) is important because of the wave shape of interface. Interfacial tension may be important especially if the curvature of the flow over the steps is sharp. Viscosity may be important if the velocity of flow is small and the viscosity high.

For the two sizes of packing there must be a difference in the fluid motion in each case depending on which two or more of the property forces are effective. This is evident also from the difference in appearance of the flow at the two sizes. Unless experimental data are obtained in the pilot plant to evaluate these force parameters in relation to scale, it will not be possible to make scale-up which allow even close approximation to dynamic similarity of motion, or to similar reaction rates.

Liquid-liquid extraction should be amenable to correlation for scale-up if the appropriate relations of Table 2 are observed. Drop size is a function of the Weber number, and is thus a function of fluid motion. Scale-up for drop and bubble size must be made with attention to the Weber number. The formation of drops of immiscible liquids due to high velocity streams such as are developed by mixing impellers in tanks, or by flow jets of liquid, is a function of interfacial tension and the Weber parameter must be taken into consideration. The relations in Table 2 require that interfacial kinematic tension must vary as some function of the scale, if dynamic similarity is to be obtained. Therefore, if the same liquids are used in two sized systems the drop size cannot bear the same relation to system size (the chosen length basis, or the scale) in each case. This no doubt accounts for the fact that drops of liquid produced by mixing impellers are of different size when produced in different sized systems operating at the same Reynolds number. The rate of settling (or rise) of drops, once formed, is a function of viscous forces and Equation (12) gives the relation for translation from pilot plant to larger scale.

Some data are in the literature on distillation in wetted wall columns (*2*) which show that the H.E.T.P. is some function of the scale. However, the data were obtained as a by-product of experimentation for a different purpose, and the kinematic viscosity of the reflux liquid was not varied (as it might be by operation at different pressure and temperature), so that it is not possible to use relations of Table 2 to test for dynamic similarity and thus compare the performance under conditions of similitude.

Similitude relations are basic to the understanding of and application of flow in jets and the fluid surrounding the jet. Behavior of jets is dependent upon kinematic force parameters and data can be extended to fluids of the same kinematic viscosity (*1, 7*) when the velocities are such that fully developed turbulence is achieved.

A particularly interesting possibility for application of similitude principles is in the fluidizing operation. When a differential equation is written describing the flow and particle distribution in a fluidized stream, and operated on as described above for the case of viscous flow, it is possible to obtain a relation like Equation (18). The parameters determined therefrom set the similitude requirements. Experimental work on fluidizing have not, to the writer's knowledge, been designed with such complete relations in mind. Reynolds numbers have of course been used as criteria, but other less evident relations can be developed, and they should be investigated. It is of interest to note that similitude relations have been successfully applied to the study of the transport and pick-up of silt in river flow. There are interesting analogies between such an operation and fluidizing.

Other interesting applications of the mechanics of similitude to fluid problems of direct interest to the chemical engineer can be found (*5, 8*).

Notation—Force Units

Mass units may be found by substituting Ma for F.

Mass, force, length, and time are related by $\frac{ML}{FT^2} = 1$

a = acceleration, L/T^2
A = area, L^2
D = diameter, L
d = diffusivity, L^2/T
e = kinematic bulk modulus of elasticity, L^2/T^2
F = force, $= Ma = ML/T^2$, or Mg
g = gravity constant, L/T^2
k = mass-transfer coefficient, L/T
K = a constant, dimensionless
K_e = dynamic bulk modulus of elasticity, F/L^2
L = length dimension, FT^2/M
L_r = scale ratio
M = mass, ft.$^2/L$, or F/g
N = revolutions per second
N_c = Cauchy number, dimensionless
N_{Fr} = Froude number, dimensionless
N_I = Newton inertia force group, dimensionless
N_{Re} = Reynolds number, dimensionless
N_{We} = Weber number, dimensionless
p = pressure intensity, F/L^2
P = power, FL/T
Q = flow, gal./min.
T = time, ML/F
u = velocity, L/T
γ = (gamma) specific weight, F/L^3
μ = (mu) dynamic viscosity, FT/L^2
ν = (nu) kinematic viscosity, L^2/T
ρ = (rho) density, ft.$^2/L^4$
σ = (sigma) interfacial or surface tension, F/L
ω = (omega) kinematic interfacial tension, L^3/T^2
a,b,c = linear dimensions
u,v,w = velocities in the x, y, z coordinate directions

Subscripts 1 and 2 refer to conditions 1 and 2.

Subscript r refers to the ratio of values at two conditions.

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Discussion

R. R. Hughes (Shell Development Co., Emeryville, Calif.): Although this paper accomplishes its main purpose of bringing out the need for dimensional analysis, there are two major points I must question. In the first place, no mention is made of thermodynamic similarity. Even though this may not be a criterion in the author's particular field of interest, some mention should be made of it, in view of the fact that thermodynamic and dynamic similarity cannot in general be met simultaneously. Secondly, I do not agree with Professor Rushton that it is impossible to scale-up while keeping the ratios between inertia forces and *three* or even more additional forces constant. For instance, scale-up can be made using a constant Reynolds number, a constant Froude number and a constant Weber number, if the surface tension and the viscosity of the fluid are both altered. Alteration of the surface tension may be more difficult than the viscosity alteration, but it should be noted that it is not impossible from a theoretical viewpoint. From a practical viewpoint the more use is made of special aids such as surface agents or viscosity producing agents, the more dubious are the results, because of extraneous effects, such as non-Newtonian behavior, time dependence of surface tension, etc.

J. H. Rushton: That's a valid observation. Correction has been made in the present form of the paper. Mention has been made that it is possible to approximate dynamically similar conditions when more than two force parameters are effective, that is, to distort the model so that you can evaluate independently the effect of viscosity, the effect of gravity, and the effect of interfacial tension. Model work on ships in towing basins is done in precisely that fashion. Now, obviously, the waves behind the ship must be supplied by energy put into the propellers of the ship and so, in ship motion, there is both a viscous force and a gravity force involved. If you use the same liquid in the same towing tank but use two different ship sizes, you cannot reproduce dynamic similarity. You should really use a liquid of higher viscosity and a larger tank. Or, since sea water is given for the large ship, then you should use a lower viscosity in the small model tank. However, by properly distorting the models so that the individual effect of the Reynolds number and the individual effect of the Froude number can be found, you can add the two effects and get the interrelation between the two. Analytically you can do so by simply equating the equations for velocity. I don't want to leave the impression that you cannot closely approximate dynamic similarity when three forces are in effect. Usually, one force predominates and the other forces may be neglected but it is up to you to determine whether they can be neglected.

R. R. Hughes: Modeling with three or more dynamic dimensionless groups can not only be done by distorted models but also, at least in theory, by actual models. For example, all three of the velocity ratios for pressure, weight, and viscosity forces in Table 2 can be set equal to each other, as follows:

$$u_r = \sqrt{L_r} = \frac{\nu_r}{L_r} = \sqrt{\frac{\omega_r}{L_r}}$$

The required change in viscosity and surface tension, is thus, respectively

$$\nu_r = L_r^{3/2}$$

$$\omega_r = L_r^2$$

If, practically speaking, these viscosity and surface tension factors can be attained, an exact model is possible.

C. R. Bartels (E. R. Squibb & Sons, New Brunswick, N. J.): Would the speaker comment on the possibility of reversing this technique in the case of a system affected by several forces and operated on several scales, and from the results of experiments determining which forces are effective?

J. H. Rushton: I'm sure you can determine the interplay between the forces by varying the scale. That is the feasible way, experimentally, I think, of determining the interplay between the different force properties.

John Happel (New York University, New York): I was interested in the discussion of Figure 2, in which it is stated that similar fluid motion is obtained with two different-sized tanks when both Reynolds and Froude numbers are maintained constant. When you showed in the moving picture the larger tank with propeller moving around that had the same Reynolds number as the smaller tank, but not the same Froude number, the indication was that there was only a slight vortex at the top of the larger tank. I could not help but be impressed by the fact that although the Reynolds number was the same in both cases, the propeller was rotating a lot more slowly in the case of the large tank. Doesn't dynamic similarity imply keeping constant both the inertia number and the Reynolds number? In that case, to get complete similarity, even assuming no vortex is formed, should you not have changed the viscosity so that the propeller was rotating at the same speed as it was in the smaller tank and also the Reynolds number was the same in the larger tank as in the smaller tank? Then you would have complete dimensional similarity but you would have had to change the viscosity to get it even in the presence of no vortex at all.

J. H. Rushton: You are correct in that the viscosity should be changed, as was shown. But speed must also be changed. The movie showed a 6-in. tank. In that tank there was a 2-in. impeller. The velocity at the tip of that impeller is πDN, where D is the diameter of the impeller and N is the rotational speed (revolutions per second). At a corresponding point in the larger tank, and if Reynolds number alone were controlling we should have one half the velocity and that velocity is πDN where D is now twice what it was before. Therefore, N must be one fourth what it was in the small tank to have the required velocity. However, Equation (12) shows what the velocity at corresponding points must be in order to achieve dynamic similarity at different size when viscosity is changed; and Equation (13) shows how viscosity must vary with size.

John Happel: It seems to me that if you want corresponding velocities the propellers should be rotating at the same speed in the large tank as in the small tank. At a point twice as far out from the radius you have a velocity twice as great in the larger tank.

J. H. Rushton: We must be careful to distinguish between speed (revolutions per second) and velocity (feet per second). In scale-up, if it is a function of viscous forces, we should have velocities at corresponding points as predicted by Equation (12). In the motion picture illustration, when water was used in tanks whose size ratio was two, then corresponding velocities would be one half in the larger tank if viscous forces alone were controlling and complete similarity would exist.

J. C. Jubin (Atlantic Refining Co., Philadelphia, Pa.): In the paper there is an example given where two different liquids are mixed in the same time, where the scale-up ratio is 4: the tank size is 15 ft. in one case and 60 ft. in the other case. By your analysis, the hp. requirement comes out 16 times as much to get equivalent mixing. What were the relative tank turnovers, or in other words, how many times the volume of the tank is pumped by the mixer?

J. H. Rushton: The volume of the larger tank is 64 times that of the smaller one, and for geometrically similar propellers, the flow from the slower and larger propeller is 16 times that of the small one; so the turnover in the large tank is one fourth that in the small one. For blending operations, there may well be a simple relation between turnover and size.

E. A. Fike (Monsanto Chemical Company, Nitro, W. Va.): In the example with increased viscosity, a deeper swirl appeared while all other conditions remained the same. If reaction rate were determined by agitation, would you expect to get increased reaction rate in a case of that type merely by increasing the viscosity?

J. H. Rushton: No, I merely meant to show that you get a different flow motion for different viscosities. Now the flow motion you get might be better for your operation, although molecular diffusivity would no doubt be lowered as viscosity increased.

D. B. Keyes (Heyden Chemical Corp., New York): Would it be feasible to change the apparent density in order to increase reaction rates by the addition of an inert solid such as sand and keeping the liquid reactants agitated?

J. H. Rushton: This is an interesting speculation. We know that solids suspended in liquids do change the fluid motion of the liquid, and it is quite possible that a "pseudo-viscosity" could be achieved and that reaction rates could be altered by the addition of inert solids. The feasibility of the technique would involve consideration of additions and separation of the inert solids, but *such would not deter consideration of* use of the idea for pilot plant work. Anything which will change fluid motion, and thus the convection currents, may well have an effect on the over-all rate of a chemical reaction.

J. J. Martin: Cannot all of this be stated rather simply by saying that two geometrically similar systems possess dynamic similarity of fluid motion when significant dimensionless force ratios such as the Reynolds, Froude, and Weber number but not including the Newton (or inertia) number are respectively the same for both systems?

J. H. Rushton: Certainly.

J. J. Martin: Isn't it true that if such numbers as the Froude, Weber, and Cauchy are not used in correlations of results on different chemical engineering processes, it is simply because in these applications gravitational, surface tension, and compressibility forces are found to have little effect on the operation?

J. H. Rushton: I doubt it. If this were so, why are these relations so often neglected? The purpose of this paper was to call attention to principles already well known but which have all too frequently been overlooked.

J. J. Martin: If, for example, surface tension is a proved variable in a certain operation, will it not appear in the results of a dimensional analysis of that operation and specifically, can it not be made to appear in the Weber number if size, velocity, and density are also significant variables?

J. H. Rushton: Yes, of course.

J. J. Martin: Isn't it true that "a coefficient of blending" just as a coefficient of heat transfer would be involved with tank diameter and diffusivity in a dimensionless group which would depend upon Reynolds number, ratio of impeller diameter to tank diameter, and any other significant, dimensionless ratios rather than upon the Newton or inertia number?

J. H. Rushton: Yes, Figure 3 and (*17*) show that. The inertia number would also be characteristic.

J. J. Martin: What justification is there for scaling-up a blending operation on the basis of equal Newton (inertia) numbers and equal time scales?

J. H. Rushton: The illustration was taken from commercial operating data and the results check the postulated theory.

J. J. Martin: Isn't it true that in the majority of agitation systems employed in industry means are taken to prevent vortexing and therefore surface waves are negligible, meaning that dynamic similarity in geometrically similar systems depends upon equality of Reynolds numbers alone?

J. H. Rushton: Most mixing operations are performed with equipment arranged to prevent vortexing, and unless this is done it is not convenient, if indeed possible, to relate the results to small-scale laboratory work. Therefore, I urge that small-scale mixing experimentation be done under these conditions and then dynamic similarity will be achieved at equal Reynolds numbers.

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