New frontiers in Langlands reciprocity

Subtitle Mathematics

Prof.Dr. Ana CARAIANI

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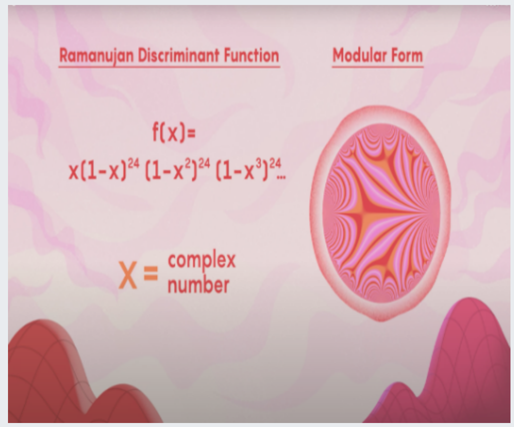
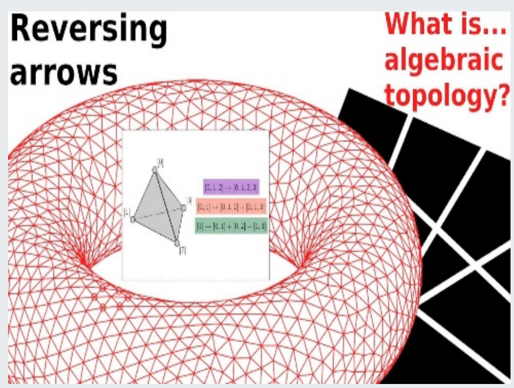
Abstract

Prof. Ana CARAIANI based her remarkable career as a mathematician on important character traits: and passion to which she added work and perseverance. The winner of 3 international mathematics Olympiads during high school, she then followed her undergraduate and doctoral studies at prestigious universities abroad. His merits were rewarded with numerous awards and distinctions at the highest international level. In this article, the author discusses some recent developments at the crossroads of arithmetic geometry and the Langlands programme. The emphasis is on recent progress on the Ramanujan–Petersson and Sato–Tate conjectures. This relies on new results about Shimura varieties and torsion in the cohomology of locally symmetric spaces.

BLOOMING
Inclusion and Diversity in STEAM





KEY TERMS

Langlands programme, arithmetic geometry, cohomology

symmetric spaces

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Introduction

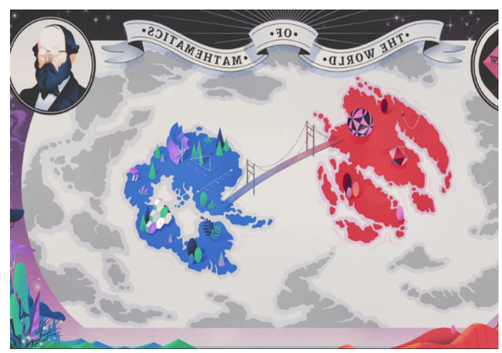
In a 1967 letter to the number theorist André Weil, a 30-year-old mathematician named Robert Langlands outlined striking conjectures that predicted a correspondence between two objects from completely different fields of math. The Langlands Programme was born.

The Langlands programme is a "grand unified theory" of mathematics: a vast network of conjectures that connect number theory to other areas of pure mathematics, such as representation theory, algebraic geometry, and harmonic analysis. One of the fundamental principles underlying the Langlands conjectures is *reciprocity*, which can be thought of as a *magical bridge* that *connects different mathematical worlds*. This principle goes back centuries to the foundational work of Euler, Legendre and Gauss on the law of quadratic reciprocity. A celebrated modern instance of reciprocity is the correspondence between modular forms and rational elliptic curves, which played a key role in Wiles's proof of Fermat's Last Theorem and which relied on the famous Taylor–Wiles method for proving modularity. Recently, the search for new reciprocity laws has begun to expand the scope of the Langlands programme. The *Ramanujan–Petersson conjecture* is an important consequence of the Langlands programme, which goes back to a prediction Ramanujan made a century ago about the size of the Fourier coefficients of a certain modular form Δ, a highly symmetric function on the upper half plane. The *Sato–Tate conjecture* is an equidistribution result about the number of points of a given elliptic curve modulo varying primes, formulated half a century ago. It is also a consequence of the Langlands programme. *Shimura varieties* are certain highly symmetric algebraic varieties that generalize modular curves and that provide, in many cases, a geometric realization of Langlands reciprocity. In Section 2, the author explain a new tool for understanding Shimura varieties called the *Hodge–Tate period morphism*. The *Calegari–Geraghty method* vastly extends the scope of the Taylor–Wiles method, though it is conjectural on an extension of the Langlands programme to incorporate torsion in the cohomology of locally symmetric spaces.

Starting from these theories, in this article Prof. Dr. Ana Caraiani formulates 17 theorems and presents 21 remarks regarding the topic of great mathematical complexity.

Methods

Presentation of the Langlands programme using a method as easy as possible to understand, for instance watching the video material realized by Rutgers University mathematician Alex Kontorovich which takes the audiences on a journey through the continents of mathematics to learn about the awe-inspiring symmetries at the heart of the Langlands programme



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Women in STEM - Facts about the author.

Prof.dr. Ana Caraiani (born 1985) is a Romanian-American mathematician, who is a Royal Society University Research Fellow and professor in the Department of Mathematics at **Imperial College London**. Her research interests include algebraic number theory and the Langlands program. In 2001, Ana Caraiani became the first Romanian female competitor in 15 years at the International Mathematical Olympiad, where she won a silver medal. In the following two years, she won two gold medals. She pursued her studies in the United States in 2003. As an undergraduate student at Princeton University, Ana Caraiani was a two-time Putnam Fellow (the only female competitor at the Putnam Mathematical Competition to win more than once). Ana Caraiani graduated *summa cum laude* from Princeton in 2007 supervised by prof. Andrew Wiles, and completed her doctoral studies at Harvard University, earning her Ph.D. in 2012 under the supervision of Wiles' student Richard Taylo. In 2007, the Association for Women in Mathematics awarded Ana Caraiani their Alice T. Schafer Prize. In 2018, she was one of the winners of the Whitehead Prize of the London Mathematical Society. She was elected as a Fellow of the American Mathematical Society in the 2020 Class. As of 2021, Ana Caraiani is a full professor at Imperial College London. She rejoined the University of Bonn in 2022 as Hausdorff Chair.

Results

- Learn about these two formerly disparate continents of math—Analytic Number Theory and Harmonic Analysis—and how a history of breakthrough thinking has started to build bridges between them.

- Learn about other important math, include math prodigy Srinivasa Ramanujan, hobbyist mathematician Pierre de Fermat, Japanese mathematicians Goro Shimura and Mikio Sato, American mathematician John Tate, German mathematician Hans Petersson, and their contributions.

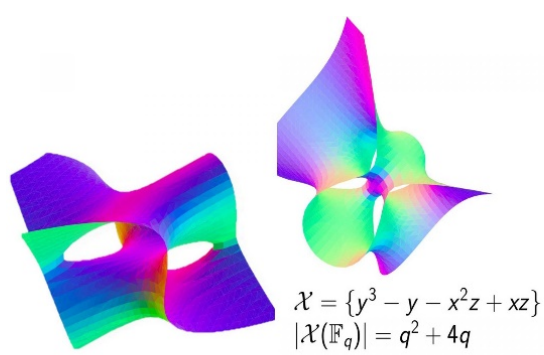
Discussion

What is the Langlands Programme?

What theory was developed by Srinivasa Ramanujan,

What theory was developed by Mikio Sato and John Tate

What is arithmetic geometry?



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Conclusion

Prod. Dr. Ana Caraian is interested in classical and p-adic Langlands programs, Shimura varieties and arithmetic geometry. The Langlands program is a beautiful but technical subject, with roots in many different areas of mathematics. In the article the author points out that for a general mathematician, Section 1 is the most accessible, as it highlights two concrete consequences of the Langlands conjectures. The later Sections 2 and 3 assume more background in algebraic geometry and number theory. Those become more difficult and require more higher level of mathematical knowledge.

Resources :

- Reference the original article: https://euromathsoc.org/magazine/articles/3

  Publish in Eur. Math. Soc. EMS Mag. 119 (2021), pp. 8–16 DOI 10.4171/MAG/3

- Any other resources that will help understanding (articles, videos, podcasts etc):
  https://www.quantamagazine.org/what-is-the-langlands-program-20220601
  https://www.quantamagazine.org/ana-caraiani-delights-in-building-mathematical-bridges-20211117/
  https://thekidshouldseethis.com/post/langlands-program-modern-mathematics-video
  https://www.google.com/search?client=firefox-b-d&q=what+is+cohomology#fpstate=ive&vld=cid:613cdfc4,vid:ol-AJVVbkMs,st:0
  https://math.mit.edu/~poonen/782/782notes.pdf
  https://thekidshouldseethis.com/post/symmetry-1961-eames-animated-math-short
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Reflection Questions:

1. **What led Prof.Dr. Ana CARAIANI to pursue a career in mathematics?**

   a. Her early curiosity in mathematics.

   b. Participation in competitions and Olympiads.

   c. Encouraging high school teachers.

   d. Parents' recommendation

   Answer: a. Her early curiosity in mathematics

2. **What are the fields of interest of Prof. Dr. Ana CARAIANI in mathematics?**

a. Probability and statistics

b. Number theory

c. Game theory

d. Mathematical logic

Answer: b. number theory

3. **What is the Langlands Programme?**

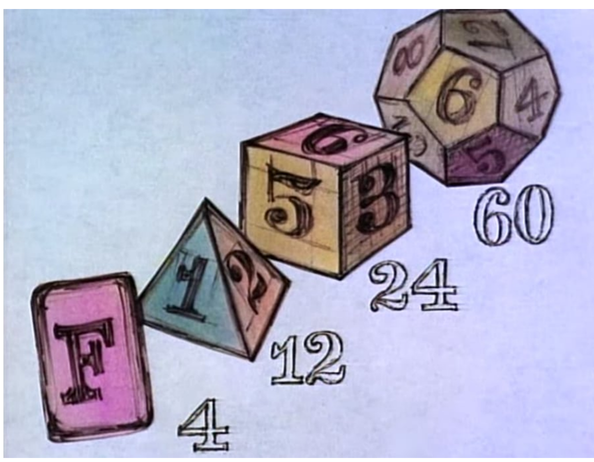
a. A mathematical theory connecting number theory, representation theory, algebraic geometry, and harmonic analysis

b. A software development project

c. A physics experiment

d. A new programming language

Answer: a. A mathematical theory connecting number theory, representation theory, algebraic geometry, and harmonic analysis



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Lesson Plan:
New Frontiers in Langlands Reciprocity

Understanding the Langlands Programme and Its Applications

Objective:

- Students will understand the basic concepts of the Langlands Programme and its significance in connecting different mathematical domains.
- Students will explore practical applications of the Langlands Programme in modern mathematics.

Materials:

- Copies of the article "New Frontiers in Langlands Reciprocity"
- Video: ["What is the Langlands Program?"]
- Whiteboard and markers
- Handouts with key terms and definitions
- Calculators
- Graph paper

Introduction (15 minutes):

1. Hook:
   - Show a short video introducing the Langlands Programme and its importance in modern mathematics.
2. Discussion:
   - Ask students what they know about the connection between different fields of mathematics.
   - Introducing the concept of the Langlands Programme.
3. Presentation:
   - Provide an overview of the article, highlighting the recent developments in the Langlands Programme, the Ramanujan-Petersson and Sato-Tate conjectures, and the significance of Shimura varieties.

Activity: Practical Applications of the Langlands Programme (30 minutes):

Reading Session (10 minutes):

1. Group Reading:
   - Have students read the introduction and key sections of the article "New Frontiers in Langlands Reciprocity" in small groups.

- Each group discusses the key points and prepares a short summary.

Exploring Mathematical Connections (20 minutes):

Practical Exercise 1: Fourier Coefficients

1. Instructions:
    - Provide students with a set of Fourier coefficients of a modular form.
    - Ask them to analyze the pattern and discuss how these coefficients relate to the Ramanujan-Petersson conjecture.

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Practical Exercise 2: Elliptic Curves and Primes

1. Instructions:
    - Give students a simple elliptic curve equation.
    - Have them compute the number of points on the curve for different prime moduli.
    - Discuss how this relates to the Sato-Tate conjecture and its equidistribution result.

Practical Exercise 3: Symmetry and Modular Forms

1. Instructions:
    - Provide examples of modular forms and ask students to identify symmetries.
    - Discuss the connection between these symmetries and Shimura varieties in the context of the Langlands Programme.

Discussion (15 minutes):

1. Group Presentations:
    - Each group presents their summary of the article and their findings from the practical exercises.
2. Q&A Session:
    - Address any questions students have about the Langlands Programme and its practical applications in modern mathematics.

Conclusion (10 minutes):

1. Summary:
    - Summarize the main points discussed during the lesson.
    - Highlight the importance of the Langlands Programme and its applications.
2. Reflection:
    - Ask students to reflect on what they have learned about the Langlands Programme and its impact on mathematics.
    - Have them write a brief paragraph on their thoughts about the use of this mathematical framework.

Assessment:

- Participation: Evaluate student participation in group discussions and presentations.
- Practical Exercises: Assess the accuracy and completeness of the practical exercises.
- Reflection Paragraph: Evaluate the students' written reflections on the Langlands Programme for understanding and insight.

Reflection Questions

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1. What is the Langlands Programme?

a. A mathematical theory connecting number theory, representation theory, algebraic geometry, and harmonic analysis

b. A software development project

c. A physics experiment

d. A new programming language

Answer: a. A mathematical theory connecting number theory, representation theory, algebraic geometry, and harmonic analysis

2. Which mathematical domains are connected by the Langlands Programme?

a. Physics and chemistry

b. Biology and astronomy

c. Number theory, representation theory, algebraic geometry, and harmonic analysis

d. Economics and sociology

Answer: c. Number theory, representation theory, algebraic geometry, and harmonic analysis

3. Who formulated the Langlands Programme?

a. Isaac Newton

b. Albert Einstein

c. Robert Langlands

d. Carl Friedrich Gauss

Answer: c. Robert Langlands

4. What is the Ramanujan–Petersson conjecture about?

a. The size of the Fourier coefficients of a certain modular form Δ

b. The behavior of prime numbers

c. The shape of the universe

d. The structure of atoms

Answer: a. The size of the Fourier coefficients of a certain modular form Δ

5.What are Shimura varieties?

a. A type of geometric shape found in nature

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b. Highly symmetric algebraic varieties that generalize modular curves

c. A new type of number discovered in the 20th century

d. A method for solving quadratic equations

Answer: b. Highly symmetric algebraic varieties that generalize modular curves