The local limit of proper supercritical percolation is uncountable

as shown by $\mathbb{P}$(the cluster of the origin is G) $= 0$ for any fixed infinite G

and moreover $\mathbb{P}$(the cluster of the origin is in $\mathcal{G}_i$) $= 0$, where $\mathcal{G}_i$ is a particular partition of connected subgraphs of the base lattice

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Let us consider bond percolation on $\mathbb{Z}^2$, and let $C_{(0,0)}$ be the random variable representing the isomorphism class of the cluster of the origin. For any rooted graph (G, o) and non-negative integer r, let $(G, o)|_r$ be the rooted graph which is the restriction of (G, o) to the vertices which are at distance at most r from the root.

Proposition 1. *Let $C_{(0,0)}$ be the cluster of the origin in Bernoulli bond percolation with parameter $p \in [0, 1)$. For any infinite rooted graph (G, o) and any non-negative integer r,*

$$\mathbb{P}\left(C_{(0,0)}|_{r+1} \simeq (G,o)|_{r+1}\right) \leq \max\left\{p, \frac{1}{\sqrt{\pi}}\right\} \mathbb{P}\left(C_{(0,0)}|_r \simeq (G,o)|_r\right), \tag{1}$$

implying

$$\mathbb{P}\left(C_{(0,0)} \simeq (G,o)\right) = 0.$$

Proof. Let an infinite rooted graph (G, o) and a non-negative integer r be fixed. The upper bound (1) for $\mathbb{P}\left(C_{(0,0)}|_{r+1} \simeq (G,o)|_{r+1}\right)$ is based on a variant of breadth-first search. We will explore the cluster of the origin of a percolation sample as follows. Suppose that prior to round $s \in \mathbb{N}$, we determined the status of some edge set $E_{s-1} \subset E(\mathbb{Z}^2)$, meaning we know exactly which edges in E_{s-1} are open and which are closed, while we have no information about the edges in $E(\mathbb{Z}^2) \setminus E_{s-1}$. We also determined all the vertices $V_s \subset V(\mathbb{Z}^2)$ which are, in the graph distance of the percolated grid, at distance at most s from the origin. It is important to note that this is a different distance than the graph distance in the original grid.

To start with, $V_0 = \{(0,0)\}$ and $E_0 = \varnothing$. We will denote by $V_{=s}$ the subset of the vertices of V_s which are at distance exactly s from $(0,0)$ in the percolated grid. In round s, we uncover all the edges whose one endpoint is in $V_{=s}$ and the other in $V(\mathbb{Z}^2) \setminus V_s$. Then E_{s+1} is the set of all the edges incident to a vertex in V_s without those whose *both* endpoints are in $V_{=i}$ for some $i \in [s]$. In fact, since $\mathbb{Z}^2$ contains no odd cycles, there can be no edges with both endpoints in $V_{=i}$ anyway, but we spell out the general approach because it can be applied to non-bipartite lattices just as successfully.

Let us suppose that prior to round $r + 1$, our exploration process yielded that $C_{(0,0)}|_r \simeq (G, o)|_r$. This means that we have uncovered an embedding H of $(G, o)|_r$ in $(\mathbb{Z}^2, (0, 0))$. Let now $e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H)$ be the number of unexplored edges with one endpoint in H and one in $\mathbb{Z}^2 \setminus H$, and let m be the number of edges between $(G, o)|_r$ and $(G, o)|_{r+1}$. Note that while $m = e\left((G,o)|_r, (G,o)|_{r+1}\right)$ only depends on the isomorphism class of (G, o) and was fixed from the get-go, $e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H)$ is a random variable and depends on the particular way $(G, o)|_r$ ended up being embedded in $(\mathbb{Z}^2, (0, 0))$.

By employing the law of total probability and Stirling's formula, we now obtain

$$\begin{aligned}
&\mathbb{P}\left(C_{(0,0)}|_{r+1} \simeq (G,o)|_{r+1} \;\middle|\; C_{(0,0)}|_r \simeq (G,o)|_r\right) \\
&\le \mathbb{P}\left(e\left(C_{(0,0)}|_r, C_{(0,0)}|_{r+1}\right) = e\left((G,o)|_r, (G,o)|_{r+1}\right) \;\middle|\; C_{(0,0)}|_r \simeq (G,o)|_r\right) \\
&= \sum_{\ell\ge 0} \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \cdot \mathbb{P}\left(\text{exactly } m \text{ edges between } H \text{ and } \mathbb{Z}^2 \setminus H \text{ are open} \;\middle|\; e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \\
&= \sum_{\ell\ge m} \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \binom{\ell}{m} p^m (1-p)^{\ell-m} \\
&= \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = m\right) p^m + \sum_{\ell\ge m+1} \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \binom{\ell}{m} p^m (1-p)^{\ell-m} \\
&\le \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = m\right) p + \sum_{\ell\ge m+1} \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \frac{\ell^\ell}{m^m(\ell-m)^{\ell-m}} \cdot \frac{1}{\sqrt{\pi}} \left(\frac{p}{1-p}\right)^m (1-p)^\ell
\end{aligned}$$

Note that in the inequality above, we used the assumption that G is infinite, and hence $m \ge 1$. Finally, by differentiating the function

$$f(m) = \frac{\ell^\ell}{m^m(\ell-m)^{\ell-m}} \cdot \left(\frac{p}{1-p}\right)^m$$

with respect to m, we see that for $m \in [1, \ell)$, its maximum is attained when $m = p\ell$, and so we can continue the chain of (in)equalities with

$$\begin{aligned}
&\le \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = m\right) p + \sum_{\ell\ge m+1} \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \frac{\ell^\ell}{(p\ell)^{p\ell}(\ell-p\ell)^{\ell-p\ell}} \cdot \frac{1}{\sqrt{\pi}} \left(\frac{p}{1-p}\right)^{p\ell} (1-p)^\ell \\
&= \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = m\right) p + \sum_{\ell\ge m+1} \mathbb{P}\left(e_{\mathrm{un}}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \cdot \frac{1}{\sqrt{\pi}} \\
&\le \max\left\{p, \frac{1}{\sqrt{\pi}}\right\}.
\end{aligned}$$

Remark 2. *In the computation above which yielded*

$$\begin{aligned}
&\mathbb{P}\left(C_{(0,0)}|_{r+1} \simeq (G,o)|_{r+1} \;\middle|\; C_{(0,0)}|_r \simeq (G,o)|_r\right) \\
&\qquad \le \mathbb{P}\left(e\left(C_{(0,0)}|_r, C_{(0,0)}|_{r+1}\right) = e\left((G,o)|_r, (G,o)|_{r+1}\right) \;\middle|\; C_{(0,0)}|_r \simeq (G,o)|_r\right) \le \max\left\{p, \frac{1}{\sqrt{\pi}}\right\},
\end{aligned}$$

we did not use the conditioning on the event $C_{(0,0)}|_r \simeq (G,o)|_r$ in any sense. It does have influence on the individual probabilities $\mathbb{P}\left(e_{un}(H, \mathbb{Z}^2 \setminus H) = \ell\right)$, but we ended up only needing

$$\sum_{\ell\ge m} \mathbb{P}\left(e_{un}(H, \mathbb{Z}^2 \setminus H) = \ell\right) \le 1,$$

and so we could have derived

$$\mathbb{P}\left(e\left(C_{(0,0)}|_r, C_{(0,0)}|_{r+1}\right) = e\left((G,o)|_r, (G,o)|_{r+1}\right)\right) \le \max\left\{p, \frac{1}{\sqrt{\pi}}\right\}$$

just as well. Nevertheless, the events $e\left(C_{(0,0)}|_i, C_{(0,0)}|_{i+1}\right) = e\left((G,o)|_i, (G,o)|_{i+1}\right)$ *are not independent for varying* i*, and so a priori we cannot bound*

$$\mathbb{P}\left(C_{(0,0)}|_{r+1} \simeq (G,o)|_{r+1}\right) \le \mathbb{P}\left(\bigcap_{i=0}^{r} e\left(C_{(0,0)}|_i, C_{(0,0)}|_{i+1}\right) = e\left((G,o)|_i, (G,o)|_{i+1}\right)\right)$$

by

$$\prod_{i=0}^{r} \mathbb{P}\left(e\left(C_{(0,0)}|_i, C_{(0,0)}|_{i+1}\right) = e\left((G,o)|_i, (G,o)|_{i+1}\right)\right) \le \left(\max\left\{p, \frac{1}{\sqrt{\pi}}\right\}\right)^{r+1}.$$

Finally,

$$\begin{aligned}\mathbb{P}\left(C_{(0,0)}|_{r+1} \simeq (G,o)|_{r+1}\right) &\le \mathbb{P}\left(e\left(C_{(0,0)}|_r, C_{(0,0)}|_{r+1}\right) = e\left((G,o)|_r, (G,o)|_{r+1}\right) \;\cap\; C_{(0,0)}|_r \simeq (G,o)|_r\right) \\ &= \mathbb{P}\left(e\left(C_{(0,0)}|_r, C_{(0,0)}|_{r+1}\right) = e\left((G,o)|_r, (G,o)|_{r+1}\right) \,\middle|\, C_{(0,0)}|_r \simeq (G,o)|_r\right) \mathbb{P}\left(C_{(0,0)}|_r \simeq (G,o)|_r\right) \\ &\le \max\left\{p, \frac{1}{\sqrt{\pi}}\right\} \mathbb{P}\left(C_{(0,0)}|_r \simeq (G,o)|_r\right)\end{aligned}$$

as was to be proven. By induction, it immediately follows that for any positive integer n,

$$\mathbb{P}\left(C_{(0,0)}|_n \simeq (G,o)|_n\right) \le \left(\max\left\{p, \frac{1}{\sqrt{\pi}}\right\}\right)^n.$$

Since both p and $\frac{1}{\sqrt{\pi}}$ are strictly less than 1, we get that $\mathbb{P}\left(C_{(0,0)}|_n \simeq (G,o)|_n\right) \to 0$ as $n \to \infty$. But

$$\mathbb{P}\left(C_{(0,0)} \simeq (G,o)\right) \le \mathbb{P}\left(C_{(0,0)}|_n \simeq (G,o)|_n\right)$$

for every n, and so necessarily

$$\mathbb{P}\left(C_{(0,0)} \simeq (G,o)\right) = 0.$$

□

Note that in the proof above, we only used the parameters $e\left((G,o)|_{i-1}, (G,o)|_i\right)$, and no other structural properties of (G,o) were needed. This in particular gives the following stronger statement. Let

$$\mathcal{G}_\bullet = \bigcup_{(m_1,m_2,\dots)\in\mathbb{N}^\mathbb{N}} \mathcal{G}_{(m_1,m_2,\dots)}$$

be a partition of the set of connected rooted infinite graphs into the classes

$$\mathcal{G}_{(m_1,m_2,\dots)} := \left\{(G,o) \;:\; e\left((G,o)|_{i-1}, (G,o)|_i\right) = m_i\right\}.$$

Then for every $(m_1, m_2, \dots) \in \mathbb{N}^\mathbb{N}$,

$$\mathbb{P}\left(C_{(0,0)} \in \mathcal{G}_{(m_1,m_2,\dots)}\right) = 0.$$

It is worth noting that for some sequences $(m_1, m_2, \dots) \in \mathbb{N}^\mathbb{N}$ of edge counts, the intersection of $\mathcal{G}_{(m_1,m_2,\dots)}$ with subgraphs of $\mathbb{Z}^2$ rooted at the origin is uncountable. E.g., $(2, 2, 2, \dots)$ is one such example.