Two notes with regards to Large Networks and Graph Limits

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February 2022

1 Proof of Proposition 14.2 which says that hypercubes are weakly norming

The proof defines a vertex partition $V(Q^d) = S_1 \cup S_2 \cup T$ and corresponding edge partition $E(Q^d) = E_0 \cup E_1 \cup E_2$. For a fixed decoration $W = \big(W_e : e \in E(Q^d)\big)$, it goes on to write out $t(Q^d, W)$ as

$$t(Q^d, W) = \int \prod_{ij \in E_0} W_{ij} \prod_{ij \in E_1} W_{ij} \prod_{ij \in E_2} W_{ij},$$

and then implicitly uses the (semi)inner product

$$\langle f, g \rangle_W = \int \left(\prod_{ij \in E_0} W_{ij} \right) f g \, dx \tag{1}$$

with the functions $f = \prod_{ij \in E_1} W_{ij}$ and $g = \prod_{ij \in E_2} W_{ij}$ to conclude by Cauchy-Schwarz that

$$t(Q^d, W) \leq \left(\int \left(\prod_{ij \in E_0} W_{ij} \right) \left(\prod_{ij \in E_1} W_{ij} \right)^2 dx \right)^{1/2} \left(\int \left(\prod_{ij \in E_0} W_{ij} \right) \left(\prod_{ij \in E_2} W_{ij} \right)^2 dx \right)^{1/2}.$$

Which is alright, but not what the proof wants – because it is stated there that

$$\int \left(\prod_{ij \in E_0} W_{ij} \right) \left(\prod_{ij \in E_1} W_{ij} \right)^2 dx = t\left(Q^d, W'\right) \text{ and } \int \left(\prod_{ij \in E_0} W_{ij} \right) \left(\prod_{ij \in E_2} W_{ij} \right)^2 dx = t\left(Q^d, W''\right) \tag{2}$$

(so that we could get to the goal of $t(Q^d, W) \leq t(Q^d, W')^{1/2} t(Q^d, W'')^{1/2}$), where W' and W'' are decorations obtained from W by symmetrising with respect to the hyperplane $x_1 = x_2$. But (2) does not hold in general, as we can see when we write the integrals out:

$$\begin{aligned}
t\left(Q^d, W'\right) &= \int \prod_{ij\in E_0} W'_{ij} \prod_{ij\in E_1} W'_{ij}(x_i,x_j) \prod_{ij\in E_2} W'_{ij}(x_i,x_j)\,dx \\
&= \int \prod_{ij\in E_0} W_{ij} \prod_{ij\in E_1} W_{ij}(x_i,x_j) \prod_{ij\in E_2} W_{\sigma(ij)}(x_i,x_j)\,dx \\
&= \int \prod_{ij\in E_0} W_{ij} \prod_{ij\in E_1} W_{ij}(x_i,x_j) \prod_{ij\in E_1} W_{ij}(x_{\sigma^{-1}(i)},x_{\sigma^{-1}(j)})\,dx \\
&\neq \int \left(\prod_{ij\in E_0} W_{ij}\right)\left(\prod_{ij\in E_1} W_{ij}\right)^2 dx
\end{aligned}$$

and similarly for W''.

I think a way to the goal $t(Q^d,W) \le t(Q^d,W')^{1/2} t(Q^d,W'')^{1/2}$ would be to use the following (semi)inner product instead of (1):

$$(U,V)_W = \int \left(\prod_{ij\in E_0} W_{ij}\right) \prod_{E_1} U_{ij}(x_i,x_j)V_{ij}(x_{\sigma(i)},x_{\sigma(j)})\,dx = \int \left(\prod_{ij\in E_0} W_{ij}\right) \prod_{E_1} U_{ij}(x_i,x_j) \prod_{E_2} V_{\sigma(ij)}(x_i,x_j)\,dx,$$

and then follow up with Cauchy-Schwarz in the form

$$t(Q^d,W) = (W',W'')_W \le (W',W')_W^{1/2}(W'',W'')_W^{1/2} = t(Q^d,W')^{1/2} t(Q^d,W'')^{1/2}.$$

2 Corollary 21.19 about local convergence in hyperfinite sequences

The Corollary says that every locally convergent hyperfinite graph sequence $(G_n)_n$ is locally-globally convergent, which is contradicted by a sequence whose odd terms are a fixed graph G, and even terms two disjoint copies of G. The proof relies on Theorem 21.16, which, however, assumes atomlessness of the limiting graphings, so the glitch is fixed just by adding to the corollary the assumption that $v(G_n) \to \infty$.